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Inertial Measurements Based Dynamic Attitude Estimation and Velocity-Free Attitude Stabilization

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Abstract— **This paper deals with the attitude estimation and control problems for rigid bodies, using inertial vector measurements. First, we revisit the attitude estimation algorithm on SO(3) that has been recently proposed in the literature, and propose some practical extensions and new insightful unit-quaternion based proofs. Then, we propose an attitude stabilization control scheme using only inertial vector measurements. The originality of this control strategy stems from the fact that the explicit reconstruction of the attitude as well as the angular velocity measurements are not required anymore.**

I. Introduction

The attitude control problem of rigid bodies has been widely studied over the last decades. The interest devoted to this problem is motivated by its technical challenges as well as its practical implications in aerospace and marine applications. The main technical difficulty encountered in this type of mechanical systems may be attributed to the fact that the orientation (angular position) of the rigid body is not a straightforward integration of the angular velocity. Nevertheless, the efforts of the research community in this field paid-off and led to a multitude of solutions to this problem, especially with the major help of the (singularity-free) unit-quaternion representation which has proven to be an instrumental tool (see, for instance, [10], [18], [23]).

As it is customary in the position control of mechanical systems, the majority of the control schemes developed for rigid bodies are (roughly speaking) of Proportional-Derivative (PD) type, where the proportional action is in terms of the orientation and the derivative action (generating the necessary damping) is in terms of the angular velocity. The requirement of the angular velocity can be removed through an appropriate design, usually based on the passivity properties of the system as done, for instance, in [3], [5], [10], [14], [18] and [22]. The explicit use of the attitude (*e.g.,* the unit-quaternion) in the control law calls for the development of suitable attitude estimation algorithms that reconstruct the attitude from the measurements provided by the appropriate sensors (depending on the type of the rigid body and its domain of application). Usually, for small scale Unmanned Aerial Vehicles (UAVs), due to cost and weight constraints, we make use of small, compact and relatively inexpensive inertial measurement units (IMUs), attached to the rigid body, equipped with three-axis accelerometers, magnetometers and gyroscopes. The gyroscopes provide the angular velocity of the rigid body, while the accelerometers and magnetometers provide, respectively, vector measurements of the acceleration and the earth magnetic field directions in the body attached frame. In the present paper, by "vector measurements" or "vector observations" we mean body-frame measurements of known vectors (or directions) in the inertial frame.

Initially, the attitude determination from vector observations, has been tackled as a static optimization problem for which several solutions, based on Wahba's problem, have been proposed [16]. These algorithms have been refined, later on, incorporating filtering techniques of Kalman-type to handle the measurement noise [17]. Extended Kalman filters have been extensively used in aerospace engineering and have proven to be the backbone of satellite attitude estimation algorithms (we refer the reader to the survey paper [4] for further details). On the other hand the most simple and yet practical dynamic IMU-based attitude estimation approach is based on linear complementary filtering [1], [19], where the vector measurements are fused with the angular velocity measurement to recover the orientation of the rigid body for small angular movements. This approach has been extended to nonlinear complementary filtering for the attitude estimation from vector measurements in [6] and [11]. The gyro bias estimation has also been addressed in [21] and incorporated in the attitude estimation algorithms developed in [11]. The IMU-based attitude estimation techniques proposed in [11] are effective for quasi-stationary flights where the linear acceleration of the body is assumed to be relatively small compared to the gravitational acceleration. This restriction has been overcome in the recent work of [7] and [12] based on the symmetry preserving observers [2], where the linear velocity of the rigid body is used together with the IMU measurements to recover the attitude in accelerated flights.

In the present paper, we revisit the vector measurement based attitude estimation algorithm, initially proposed in [6] and [11], to which we bring some additional practical insights and new proofs using unit-quaternion. On the other hand, as explained earlier, most of the existing dynamic estimation algorithms (for small scale UAVs) make use of the whole IMU's information to reconstruct the orientation of the rigid body. Therefore, a natural question that may arise is whether it makes sense to use velocity-free attitude controllers such as those proposed, for instance, in [10], [18] and [22], since the angular velocity will be used (anyways) to recover the attitude via an appropriate

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IMU-based estimation algorithm. In this context, the main contribution of the present paper, is the development of a new attitude stabilization control scheme that uses explicitly vector measurements without requiring (either directly or indirectly) the velocity measurement. This controller is a “true velocity-free ” scheme since neither the velocity nor the unit-quaternion (representing the body’s orientation) are used in the control law.

II. BACKGROUND

A. Equations of motion

In this work, we consider a rigid body whose rotational dynamics are governed by

$$\Sigma_R : \begin{cases} \dot{Q} = \frac{1}{2} Q \odot \bar{\omega}, \\ I_f \dot{\omega} = \tau - S(\omega) I_f \omega, \end{cases} \quad (1)$$

where the vector $\bar{\omega}$ is the quaternion associated to three-dimensional vector ω, denoted by $(0, \omega)$, where ω denotes the angular velocity of the rigid body expressed in the body-attached frame $\mathcal{B}$. $I_f \in \mathbb{R}^{3\times 3}$ is a symmetric positive definite constant inertia matrix of rigid body with respect to $\mathcal{B}$. The external torque applied to the system expressed in $\mathcal{B}$ is denoted by τ.

The unit quaternion $Q = (q_0, q)$, composed of a vector component $q \in \mathbb{R}^3$ and a scalar component $q_0 \in \mathbb{R}$, represents the orientation of the inertial frame $\mathcal{I}$ with respect to the body-attached frame $\mathcal{B}$, and are subject to the constraint $q^T q + q_0^2 = 1$. The rotation matrix, related to the unit-quaternion Q, that brings the inertial frame into the body-attached frame, can be obtained through the Rodriguez formula as

$$R(Q) = (q_0^2 - q^T q) I_3 + 2qq^T - 2q_0 S(q) \quad (2)$$

where I_3 is the 3-by-3 identity matrix and $S(x)$ is the skew-symmetric matrix associated to the vector $x \in \mathbb{R}^3$ such that $S(x)V = x \times V$ for any vector $V \in \mathbb{R}^3$, where $\times$ denotes the vector cross product.

B. Quaternion preliminaries

The set of quaternion $\mathbb{Q}$ is a four-dimensional vector space over the reals, which forms a group with the quaternion multiplication denoted by “$\odot$”. The quaternion multiplication is distributive and associative but not commutative [15]. The multiplication of two quaternion $P = (p_0, p)$ and $Q = (q_0, q)$ is defined as

$$P \odot Q = (p_0 q_0 - p^T q\ ,\ p_0 q + q_0 p + p \times q), \quad (3)$$

and has the quaternion $(1, \mathbf{0})$ as the identity element. Note that, for a given quaternion $Q = (q_0, q)$, we have $Q \odot Q^{-1} = Q^{-1} \odot Q = (1, \mathbf{0})$, where $Q^{-1} = \frac{(q_0, -q)}{\|Q\|^2}$.

Note that in the case where $Q = (q_0, q)$ is a unit-quaternion, the inverse is given by $Q^{-1} = (q_0, -q)$.

Throughout this paper, we will denote by $\bar{X} := (0, X)$ the quaternion associated to the three-dimensional vector X. A vector $x_{\mathcal{I}}$ expressed in the inertial frame $\mathcal{I}$ can be expressed in the body frame $\mathcal{B}$ by $x_{\mathcal{B}} = R x_{\mathcal{I}}$ or equivalently in terms of unit-quaternion as $\bar{x}_{\mathcal{B}} = Q^{-1} \odot \bar{x}_{\mathcal{I}} \odot Q$, where $\bar{x}_{\mathcal{I}} = (0, x_{\mathcal{I}})$, $\bar{x}_{\mathcal{B}} = (0, x_{\mathcal{B}})$, and Q is the unit-quaternion associated to R as per (2).

III. ATTITUDE ESTIMATION

A. Attitude estimation using raw vector measurements

In this section, we derive an attitude estimation algorithm that relies on set of vector measurements in the body-attached frame associated to a set of known inertial measurement. We assume that the angular velocity as well as the vector measurements are known.

Consider $n \geq 2$ measured vectors b_i in the body attached frame, corresponding to n known inertial vectors r_i such that $b_i = R r_i$ or equivalently in terms of the unit-quaternion

$$\bar{b}_i = Q^{-1} \odot \bar{r}_i \odot Q. \quad (4)$$

Assume that we have two non-collinear vectors among the measured vectors. We consider the following unit-quaternion based attitude observer

$$\dot{\hat{Q}} = \frac{1}{2} \hat{Q} \odot \bar{\beta}, \quad (5)$$

$$\beta = \omega - \sum_{i=1}^{i=n} \gamma_i S(\hat{b}_i) b_i, \quad (6)$$

where $\gamma_i > 0$, and $\hat{b}_i$ is the vector part of $\bar{\hat{b}}_i = \hat{Q}^{-1} \odot \bar{r}_i \odot \hat{Q}$.

Note that this attitude observer has been proposed in [6], [11] in terms of the rotation matrix instead of the unit-quaternion. Here the result of [6] is revisited and an alternative proof, in terms of the quaternion, is provided. In the sequel, we will denote the orientation error by $\tilde{R} = \hat{R}^T R$ which corresponds to the quaternion error $\tilde{Q} = Q \odot \hat{Q}^{-1}$.

Our result is stated in the following proposition.

Proposition 1: Consider the observer (5)-(6), with $n \geq 2$ vector measurements b_i, corresponding to the inertial vectors r_i, $i = 1, \ldots, n$. Assume that there are at least two non-collinear vectors among the n inertial vectors. Assume that the angular velocity ω is bounded. Then

- *i)* $\lim_{t\to\infty} \tilde{R}(t) = I$ (equivalently $\lim_{t\to\infty} \tilde{Q}(t) = (\mathrm{sgn}(\tilde{q}_0(0)), 0)$), for almost any initial condition (*i.e.,* except for a set of Lebesgue measure zero described by $\Psi = \{\tilde{Q} = (\tilde{q}_0, \tilde{q}) \in S^3 \mid \tilde{q}_0 = 0\}$).
- *ii)* The set Ψ is forward invariant and non-attractive.
- *iii)* If $\tilde{q}_0(0) = 0$, $\tilde{Q}(t)$ will converge to one of the equilibria $\tilde{Q} = (0, \pm\tilde{q})$, where $\tilde{q}$ are the unit eigenvectors of the matrix $M = \sum_{i=1}^{n} \gamma_i r_i r_i^T$.

Proof: Consider the following Lyapunov function candidate

$$V = \frac{1}{2} \sum_{i=1}^{n} \gamma_i \tilde{b}_i^T \tilde{b}_i, \quad (7)$$

where $\tilde{b}_i$ is the vector part of the quaternion

$$\bar{\tilde{b}}_i = \bar{\hat{b}}_i - \bar{b}_i. \quad (8)$$

The time derivative of (7), in view of (1) and (5), is given by

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n} \gamma_i \tilde{b}_i^T (S(\hat{b}_i)\beta - S(b_i)\omega) \\ &= \sum_{i=1}^{n} \gamma_i \tilde{b}_i^T (S(\hat{b}_i)\beta - S(\hat{b}_i)\omega + S(\tilde{b}_i)\omega) \\ &= \sum_{i=1}^{n} \gamma_i \tilde{b}_i^T S(\hat{b}_i)(\beta - \omega) \end{aligned} \quad (9)$$

where we used the fact that

$$\frac{d}{dt}\bar{\hat{b}}_i = \frac{d}{dt}\left(\hat{Q}^{-1} \odot \bar{r}_i \odot \hat{Q}\right) = (0, S(\hat{b}_i)\beta), \quad (10)$$

and

$$\frac{d}{dt}\bar{b}_i = \frac{d}{dt}\left(Q^{-1} \odot \bar{r}_i \odot Q\right) = (0, S(b_i)\omega). \quad (11)$$

Using (6) and the fact that $S(\hat{b}_i)b_i = -S(\hat{b}_i)\tilde{b}_i$, the Lyapunov time-derivative (9), becomes

$$\dot{V} = -(\sum_{i=1}^{n} \gamma_i S(\hat{b}_i)b_i)^T (\sum_{i=1}^{n} \gamma_i S(\hat{b}_i)b_i), \quad (12)$$

It is clear that all signals involved in the control scheme are bounded, and V is nonincreasing and converges to a constant as t goes to infinity. Due to the boundedness of $\ddot{V}$, one can conclude that

$$\lim_{t\to\infty} \sum_{i=1}^{n} \gamma_i S(\hat{b}_i)b_i = 0. \quad (13)$$

One can show that

$$S(\hat{b}_i)b_i = \hat{R}S(r_i)\tilde{R}r_i \quad (14)$$

with $\tilde{R} = \hat{R}^T R$, where R and $\hat{R}$ are the rotation matrices associated, respectively, to Q and $\hat{Q}$. Therefore, in view of (14), equation (13) leads to

$$\lim_{t\to\infty} \sum_{i=1}^{n} \gamma_i (r_i \times \tilde{R}(t)r_i) = 0. \quad (15)$$

It is clear, from (15), that if all vectors are collinear, $\tilde{R}$ would be any rotation matrix about the axis collinear to r_i. Therefore, at least two non-collinear vector measurements are required to avoid this situation. In this case, it is clear that $\tilde{R} = I_3$ (or equivalently $\tilde{Q} = (\pm 1, 0)$) is an equilibrium which is the desired one. However, there are other "undesired" equilibria as it will be shown next.

Using the unit quaternion $\tilde{Q}$ corresponding to $\tilde{R}$, the equilibrium equation $\sum_{i=1}^{n} \gamma_i (r_i \times \tilde{R}r_i) = 0$, leads to

$$\sum_{i=1}^{n} \gamma_i S(r_i)((\tilde{q}_0^2 - \tilde{q}^T\tilde{q})I_3 + 2\tilde{q}\tilde{q}^T - 2\tilde{q}_0 S(\tilde{q}))r_i = 0. \quad (16)$$

which reduces to

$$\sum_{i=1}^{n} \gamma_i S(r_i)(\tilde{q}\tilde{q}^T - \tilde{q}_0 S(\tilde{q}))r_i = 0. \quad (17)$$

Multiplying the previous equation by $\tilde{q}^T \neq 0$, and using the properties of the skew symmetric matrix, one gets

$$-\tilde{q}_0 \tilde{q}^T W \tilde{q} = 0. \quad (18)$$

where $W = -\sum_{i=1}^{n} \gamma_i S(r_i)^2 = \sum_{i=1}^{n} \gamma_i (r_i^T r_i I - r_i r_i^T)$. Since W is positive definite as long as we have at least two non-collinear vectors r_i, it is clear that the equilibria are characterized by $\tilde{q}_0 = 0$. In this case, we have "undesired" equilibria given by $\tilde{Q} = (0, \pm\tilde{q})$, where $\tilde{q}^T\tilde{q} = 1$. In fact, in the case where $\tilde{q}_0 = 0$, the equilibrium equation $\sum_{i=1}^{n} \gamma_i (r_i \times \tilde{R}r_i) = 0$ is equivalent to $\sum_{i=1}^{n} \gamma_i S(r_i)\tilde{q}\tilde{q}^T r_i = 0$, which can be written as

$$S(\tilde{q})M\tilde{q}\tilde{q}^T = 0, \quad (19)$$

with $M = \sum_{i=1}^{n} \gamma_i r_i r_i^T$. Multiplying (19) by $\tilde{q} \neq 0$ and using the fact that $\tilde{q}^T\tilde{q} = 1$, one gets

$$S(\tilde{q})M\tilde{q} = 0, \quad (20)$$

which shows that, the vector parts $\tilde{q}$ of the "undesired" equilibria are the unit eigenvectors of M.

Now, let us show the set Ψ is forward invariant and unstable (non-attractive).

The time derivative of the quaternion error $\tilde{Q} = Q \odot \hat{Q}^{-1}$ corresponding to $\tilde{R}$, is given by

$$\dot{\tilde{Q}} = \frac{1}{2}\tilde{Q} \odot \bar{\tilde{\omega}} \quad (21)$$

where $\tilde{\omega} = \hat{R}^T(\omega - \beta)$. In particular, in view of (6) we have

$$\begin{aligned} \dot{\tilde{q}}_0 &= -\tfrac{1}{2}\tilde{q}^T\tilde{\omega} = -\tfrac{1}{2}\tilde{q}^T\hat{R}^T \sum_{i=1}^{n} \gamma_i S(\hat{b}_i)b_i \\ &= -\tfrac{1}{2}\tilde{q}^T\hat{R}^T \sum_{i=1}^{n} \gamma_i \hat{R}S(r_i)\tilde{R}r_i \\ &= -\tfrac{1}{2}\tilde{q}^T \sum_{i=1}^{n} \gamma_i S(r_i)\tilde{R}r_i \\ &= -\tilde{q}^T \sum_{i=1}^{n} \gamma_i S(r_i)(\tilde{q}\tilde{q}^T - \tilde{q}_0 S(\tilde{q}))r_i \\ &= \tilde{q}_0 \tilde{q}^T W \tilde{q}, \end{aligned} \quad (22)$$

Therefore, it is clear that $\dot{\tilde{q}}_0 = 0$ for $\tilde{q}_0 = 0$, this shows the invariance of the set Ψ. One can also show that

$$\frac{d}{dt}(\tilde{q}_0^2) = 2\tilde{q}_0^2\tilde{q}^T W \tilde{q} \geq 0, \quad (23)$$

which shows that $|\tilde{q}_0|$ is non-decreasing and hence if $\tilde{q}_0(0) \neq 0$, $\tilde{q}_0(t)$ will never cross zero for all $t \geq 0$, and this shows that the set Ψ is a repeller. ■

B. Attitude estimation using filtered vector measurements

In practical applications, the measurements are often contaminated with noise. A common practice, well known by control engineers, is to introduce low-pass filters and use the filtered signals in the estimation and control algorithms. This filtering procedure is routinely used in control applications, often without any rigorous proof of stability.

In this section, instead of using raw vector measurements in the estimation algorithm of Proposition 1, we suggest the use of filtered vector measurements. The filter parameters can be selected by the designer to set up the desired noise cut-off. The overall closed loop stability results of the new scheme remain similar to those of proposition 1. Moreover, as it will become clear later, the additional filter brings in an extra degree of freedom, instrumental in *breaking* the singularity at the undesired equilibrium characterized by $\tilde{q}_0 = 0$ through an adequate choice of the filter's initial conditions. Our new attitude observer, using filtered vector measurements, is given in the following proposition:

Proposition 2: Consider the rotational dynamics (1) with the following observer

$$\dot{\hat{Q}} = \frac{1}{2}\hat{Q} \odot \bar{\beta}, \tag{24}$$

with $\beta = \omega - \alpha\psi$ and ψ is given by

$$\dot{\psi} = -\alpha\psi + \alpha\sum_{i=1}^{i=n}\gamma_i S(\hat{b}_i)b_i, \tag{25}$$

The vector $\hat{b}_i$ is the vector part of $\bar{\hat{b}}_i = \hat{Q}^{-1}\odot\bar{r}_i\odot\hat{Q}$. Assume that there are at least two non-collinear vectors among the $n \geq 2$ inertial vectors. Assume also that the angular velocity ω is bounded. Then, there exists $\alpha > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$, such that

i) The estimator has the following equilibria: $(\tilde{q}_0 = \pm 1, \tilde{q} = 0, \psi = 0)$ and $(\tilde{q}_0 = 0, \tilde{q} = v, \psi = 0)$, where v are the unit eigenvectors of M.

ii) The equilibria $(\tilde{q}_0 = \pm 1, \tilde{q} = 0, \psi = 0)$ are almost globally asymptotically stable and the equilibria $(\tilde{q}_0 = 0, \tilde{q} = v, \psi = 0)$ are unstable.

Proof: Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\sum_{i=1}^{n}\gamma_i\tilde{b}_i^T\tilde{b}_i + \frac{1}{2}\psi^T\psi, \tag{26}$$

The time derivative of (26), in view of (1), (24), is given by

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{n}\gamma_i\tilde{b}_i^T(S(\hat{b}_i)\beta - S(b_i)\omega) + \psi^T\dot{\psi} \\ &= \sum_{i=1}^{n}\gamma_i\tilde{b}_i^T(S(\hat{b}_i)\beta - S(\hat{b}_i)\omega + S(\tilde{b}_i)\omega) + \psi^T\dot{\psi} \\ &= \sum_{i=1}^{n}\gamma_i\tilde{b}_i^T S(\hat{b}_i)(\beta - \omega) + \psi^T\dot{\psi} \\ &= (\beta - \omega)^T\sum_{i=1}^{n}\gamma_i S(\hat{b}_i)b_i + \psi^T\dot{\psi}\end{aligned} \tag{27}$$

Using (25), the Lyapunov time-derivative (27), becomes

$$\dot{V} = -\alpha\psi^T\psi, \tag{28}$$

It is clear that all signals involved in the control scheme are bounded, and V is nonincreasing and converges to a constant as t goes to infinity. One can also show that $\ddot{V}$ is bounded, and hence, one can conclude that $\lim_{t\to\infty}\psi(t) = 0$. Since the right hand side of (25) is uniformly continuous and $\lim_{t\to\infty}\psi(t) = 0$, it is clear that

$$\lim_{t\to\infty}\sum_{i=1}^{n}\gamma_i S(\hat{b}_i)b_i = 0. \tag{29}$$

Using the same arguments of the proof of Proposition 1, one can show that $\lim_{t\to\infty}\tilde{Q}(t) = (\pm 1, 0)$, or $\lim_{t\to\infty}\tilde{Q}(t) = (0, \pm v)$, where v are the unit eigenvectors of M. Therefore, the closed equilibria are given by

a) Desired equilibria: $(\tilde{q}_0 = \pm 1, \tilde{q} = 0, \psi = 0)$
b) Undesired equilibria: $(\tilde{q}_0 = 0, \tilde{q} = v, \psi = 0)$, where v are the unit eigenvectors of M.

Now, let us show that the equilibrium point $(\tilde{q}_0 = 0, \tilde{q} = v, \psi = 0)$ is unstable using Chetaev arguments [9]. Let us define $\delta \equiv \tilde{q}^T\hat{R}^T\psi$, and consider the dynamics of $\tilde{q}_0$ and δ around the equilibrium point $(\tilde{q}_0 = 0, \tilde{q} = v, \psi = 0)$, where the quadratic term in ψ has been discarded

$$\begin{aligned}\dot{\tilde{q}}_0 &= -\tfrac{\alpha}{2}\delta \\ \dot{\delta} &= -\alpha\delta - 2\alpha\eta\tilde{q}_0 + v^T\hat{R}^T S(\omega)\psi\end{aligned} \tag{30}$$

where $\eta = v^T W v$ and v is an eigenvector of M (*i.e.,* v is the value of $\tilde{q}$ when $\tilde{q}_0 = 0$). Consider the Chetaev function

$$\mathcal{V} = -\tilde{q}_0\delta$$

whose time derivative, in view of (30), is given by

$$\begin{aligned}\dot{\mathcal{V}} &= \tfrac{\alpha}{2}\delta^2 + 2\alpha\eta\tilde{q}_0^2 + \alpha\tilde{q}_0\delta - \tilde{q}_0 v^T\hat{R}^T S(\omega)\psi \\ &\geq \tfrac{\alpha}{2}\delta^2 + 2\alpha\eta\tilde{q}_0^2 - \alpha(\epsilon_1\tilde{q}_0^2 + \tfrac{\delta^2}{4\epsilon 1}) \\ &\quad -\|\omega\|(\epsilon_2\tilde{q}_0^2 + \tfrac{\kappa^2\delta^2}{4\epsilon 1}) \\ &\geq k_1\delta^2 + k_2\tilde{q}_0^2\end{aligned} \tag{31}$$

where $k_1 = \frac{\alpha}{2} - \frac{\alpha}{4\epsilon_1} - \frac{k_\omega\kappa^2}{4\epsilon_2}$ and $k_2 = 2\alpha\eta - \alpha\epsilon_1 - k_\omega\epsilon_2$, and k_ω is the upper bound of ω, *i.e.,* $||\omega(t)|| \leq k_\omega$. We also used the fact that ψ and δ are bounded and $\delta \neq 0$ for $\psi \neq 0$, which guarantees the existence of a finite gain $\kappa > 0$ such that $\|\psi\| \leq \kappa|\delta|$. Note that Young's inequality has been used, with arbitrary $\epsilon_1 > 0$ and $\epsilon_2 > 0$, to obtain the result in (31). Pick η sufficiently large such that $k_1 > 0$ and $k_2 > 0$.

Define the set

$$B_r = \{x \equiv (\tilde{q}_0, \delta) \in [-1\ ,\ 1] \times \mathbb{R} \mid \|x(t)\| < r\}$$

where $0 < r < 1$. Note that $\dot{\mathcal{V}} > 0$ on B_r. Let us also define a subset of B_r where $\mathcal{V} > 0$, that is

$$U_r = \{x \in B_r \mid \mathcal{V}(x) > 0\}$$

Note that U_r is non-empty for all $0 < r < 1$. Pick the initial conditions, around the equilibrium point, such that $x(0) \in U_r$ and $\mathcal{V}(x(0)) = \sigma > 0$. It is clear that $x(t)$ must leave U_r since $\mathcal{V}(x)$ is bounded on U_r and $\dot{\mathcal{V}}(x) > 0$ everywhere in U_r. Since $\mathcal{V}(x(t)) \geq \sigma$, it is clear that $x(t)$ must leave U_r through the circle $\|x\| = r$ and not through the edges $\mathcal{V}(x) = 0$ (*i.e.,* $\delta = 0$ or $\tilde{q}_0 = 0$). Since this can happen for arbitrarily small r, such as the liberalization is valid, it is clear that $(\tilde{q}_0 = 0, \delta = 0)$ is an unstable equilibrium. ∎

Remark 1: The result of Proposition 2 can also be looked at from the passivity point of view. In fact, it is clear (from the proof of Proposition 2 that the mapping from $(\beta - \omega)$ to $\sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i$ is passive. Knowing that the passivity is preserved for a passive system in cascade with a Strictly Positive Real (SPR) transfer function, one can take β in Proposition 2 as $\beta = \omega - \alpha\psi$, and $[\psi] = H(s)\left[\sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i\right]$, where $H(s)$ is any SPR filter.

Remark 2: It is clear that the main advantage of the filter, besides cleaning the measurements noise, is to relax the restriction on the initialization at $\tilde{q}_0 = 0$ through an appropriate choice of the initial conditions of the filter $\psi(0)$. In fact, it is clear that if $\tilde{q}(0)^T \hat{R}(0)^T \psi(0) \neq 0$, the manifold $\Psi = \{\tilde{Q} = (\tilde{q}_0, \tilde{q}) \in S^3 \mid \tilde{q}_0 = 0\})$ is not invariant, which causes the estimator trajectories, initialized on this set to leave it to ultimately reach the desired equilibrium.

IV. Velocity-free attitude stabilization using vector measurements

In this section, we assume that the angular velocity ω is not available for feedback. We assume that we have $n \geq 2$ measured vectors b_i in the body attached frame, corresponding to n known inertial vectors r_i such that $b_i = Rr_i$ or equivalently in terms of the unit-quaternion as given by (4). We assume that among the n measured vectors, at least two are non-collinear.

Our objective is to design a control law to stabilize the attitude of the rigid body (*i.e.,* $\lim_{t\to\infty} R(t) = I$, or $\lim_{t\to\infty} Q(t) = (\pm 1, 0)$) using only vector measurements without any knowledge of the angular velocity. Note that the attitude control of a rigid body using vector measurements, in the case where a biased angular velocity is available, has been dealt with in [13].

Let $\tilde{b}_i = \hat{b}_i - b_i$, with $\hat{b}_i$ being the vector part of $\bar{\hat{b}}_i = \hat{Q}^{-1} \odot \bar{r}_i \odot \hat{Q}$, where $\hat{Q}$ is the unit quaternion generated by the following auxiliary system

$$\dot{\hat{Q}} = \frac{1}{2}\hat{Q} \odot \bar{\beta}, \tag{32}$$

with $\hat{Q}(0) = (\hat{q}_0(0), \hat{q}(0))$ being any arbitrary unit-quaternion, and the input β being

$$\beta = -\sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i. \tag{33}$$

with $\gamma_i > 0$. we propose the following angular velocity-free control law:

$$\tau = -\sum_{i=1}^{n} S(b_i)(\rho_i r_i + \gamma_i \hat{b}_i), \tag{34}$$

with $\rho_i > 0$. Now, we can state the following result:

Theorem 1: Consider system (1) under the control law (34). Assume that we have $n \geq 2$ vector measurements b_i, corresponding to the inertial vectors r_i, $i = 1, \ldots, n$, and that there are at least two non-collinear vectors among the n inertial vectors. Then, there exists strictly positive gains ρ_i and γ_i, $i = 1, .., n$, such that all signals are bounded and $\lim\limits_{t\to\infty} \omega(t) = 0$, and $\lim\limits_{t\to\infty} Q(t) = (\pm 1, 0)$ for almost all initial condition excluding the manifold $\Psi_q = \{Q = (q_0, q) \in S^3 \mid q_0 = 0\}$.

Proof: Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\sum_{i=1}^{n} \gamma_i \tilde{b}_i^T \tilde{b}_i + \frac{1}{2}\sum_{i=1}^{n} \rho_i (b_i - r_i)^T (b_i - r_i) + \frac{1}{2}\omega^T I_f \omega \tag{35}$$

whose time-derivative, in view of (1), (10) and (11) is given by

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} \gamma_i \tilde{b}_i^T S(\hat{b}_i)(\beta - \omega) \\
&\quad + \sum_{i=1}^{n} \rho_i (b_i - r_i)^T S(b_i)\omega + \omega^T \tau \\
&= \sum_{i=1}^{n} \gamma_i \tilde{b}_i^T S(\hat{b}_i)(\beta - \omega) - \sum_{i=1}^{n} \rho_i r_i^T S(b_i)\omega + \omega^T \tau \\
&= (\beta - \omega)^T \sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i \\
&\quad + \omega^T \sum_{i=1}^{n} \rho_i S(b_i) r_i + \omega^T \tau
\end{aligned} \tag{36}$$

which in view of (33) and (34), leads to

$$\dot{V} = -(\sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i)^T (\sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i), \tag{37}$$

Consequently, it is clear that ω, $\tilde{b}_i$, $(b_i - r_i)$ are bounded (the measurements b_i as well as the the estimates $\hat{b}_i$ are naturally bounded since the inertial directions r_i are bounded. The unit quaternion Q and $\hat{Q}$ are bounded by definition. Hence, all signals involved in the control scheme are bounded. It is clear that V is nonincreasing and converges to a constant as t goes to infinity. Due to the boundedness of $\ddot{V}$, one can conclude that $\lim_{t\to\infty} \sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i = 0$. Therefore, from (33), it is clear that $\lim_{t\to\infty} \beta(t) = 0$. From the fact that $\lim_{t\to\infty} \sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i = 0$, using the same arguments of the proof of Proposition 1, one can show that $\lim_{t\to\infty} \tilde{Q}(t) = (\pm 1, 0)$, or $\lim_{t\to\infty} \tilde{Q}(t) = (0, \pm v)$, where v are the unit eigenvectors of M. Furthermore, one can show that $\ddot{\tilde{Q}}$ is bounded, and hence, $\lim\limits_{t\to\infty} \dot{\tilde{Q}}(t) = 0$, which in view of (21), implies that $\lim\limits_{t\to\infty} (\omega(t) - \beta(t)) = 0$. Consequently, $\lim\limits_{t\to\infty} \omega(t) = 0$ since $\lim_{t\to\infty} \beta(t) = 0$. One can also show that $\ddot{\omega}$ is bounded, and hence, the fact that $\lim\limits_{t\to\infty} \omega(t) = 0$, implies that $\lim\limits_{t\to\infty} \dot{\omega}(t) = 0$. Consequently, from (1), it follows that $\tau(t)$ tends to zero as t goes to infinity. Using this last fact, together with the fact that $\lim_{t\to\infty} \sum_{i=1}^{n} \gamma_i S(\hat{b}_i) b_i = 0$, one can conclude from (34) that $\lim_{t\to\infty} \sum_{i=1}^{n} \rho_i S(b_i) r_i = 0$. Again, using the same arguments of the proof of Proposition 1, one can show that $\lim_{t\to\infty} Q(t) = (\pm 1, 0)$, or $\lim_{t\to\infty} Q(t) = (0, \pm v_c)$, where v_c are the unit eigenvectors of $M_c = \sum_{i=1}^{n} \rho_i r_i r_i^T$. Now, let us show that the set Ψ_q is non-attractive. In fact, the dynamics of q_0 are given by $\dot{q}_0 = -\frac{1}{2}q^T\omega$. Around the equilibrium points, we know that $(\omega - \beta)$ tends to zero and hence $\omega \simeq \beta = -\sum_{i=1}^{n}\gamma_i S(\hat{b}_i)b_i$. We also know that τ tends to zero which, in view of (34), allows to conclude that $\sum_{i=1}^{n}\gamma_i S(\hat{b}_i)b_i$ tends to $\sum_{i=1}^{n}\rho_i S(b_i)r_i$. Consequently, the asymptotic dynamics around the equilibrium points is given by $\dot{q}_0 = q_0 q^T W_c q$, with $W_c = -\sum_{i=1}^{n}\rho_i S(r_i)^2$. This rough analysis, shows that $\Psi_q = \{Q = (q_0, q) \in S^3 \mid q_0 = 0\}$ is asymptotically non-attractive, since $|q_0|$ is non-decreasing around the equilibrium point. More rigorously, one can show this fact using Chetaev's theorem as in the proof of Proposition 2. This part has been omitted due to space limitation. ■

Remark 3: It is worth noting that if the system trajectories are initialized in Ψ_q (*i.e.,* $q_0(0) = 0$) and $q(0)^T\omega(0) \neq 0$, the system trajectories will leave the manifold Ψ_q to ultimately converge to the desired equilibria ($q_0 = \pm 1, q = 0, \omega = 0$). This is clear from the fact that $\dot{q}_0 = -\frac{1}{2}q^T\omega$ and the equilibria characterized by $q_0 = 0$ are unstable.

Remark 4: In practical applications involving small scale VTOL-UAVs, for instance, it is customary to equip the vehicle with an inertial measurement unit (IMU) composed of accelerometers, magnetometers and gyroscopes. The gyroscopes provide the angular velocity ω, the magnetometers provide a vector measurement of the earth magnetic field in the body attached frame $m_{\mathcal{B}}$, which is related to the earth's magnetic field $m_{\mathcal{I}}$ expressed in the inertial frame through $\bar{m}_{\mathcal{B}} = Q^{-1} \odot \bar{m}_{\mathcal{I}} \odot Q$. The accelerometers provide a vector measurement of the acceleration $a_{\mathcal{B}}$ in the body attached frame, which is related to the acceleration $a_{\mathcal{I}}$ expressed in the inertial frame through $\bar{a}_{\mathcal{B}} = Q^{-1} \odot \bar{a}_{\mathcal{I}} \odot Q$. In the case of quasi-stationary flights (*i.e.,* $||\dot{v}|| \ll g$), the acceleration expressed in the inertial frame is given by $a_{\mathcal{I}} = -ge_3$. The estimation algorithms of Proposition 1 and Proposition 2, could be applied directly using raw measurement obtained from the IMU $(\omega, a_{\mathcal{B}}, m_{\mathcal{B}})$, taking $n = 2$, $r_1 = a_{\mathcal{I}} = -ge_3$, $b_1 = a_{\mathcal{B}}$, $r_2 = m_{\mathcal{I}}$ and $b_2 = m_{\mathcal{B}}$. In the case of Theorem 1, the gyroscopes are not necessary and the attitude stabilization controller could be implemented using only accelerometers measurements $b_1 = a_{\mathcal{B}}$ and magnetometers measurements $b_2 = m_{\mathcal{B}}$.

V. Conclusion

In existing velocity-free attitude control schemes, the orientation (*e.g.,* unit-quaternion) appears explicitly in the control law. Therefore, in UAV applications, where the attitude is obtained via IMU-measurements-based estimation algorithms that require the angular velocity, the existing velocity-free attitude controllers do not make much sense. Motivated by this fact, we proposed a velocity-free attitude stabilization scheme that does not require the orientation reconstruction. In fact, the vector measurements are directly incorporated in the control scheme. The direct use of vector measurements in the control scheme has been initiated in [13], using gyro measurements with constant bias estimation. To the best of our knowledge, the proposed control scheme is the first incorporating directly vector measurements without any knowledge of the angular velocity.

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