

# Adaptive synchronization of networked Lagrangian systems with irregular communication delays

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**Abstract**—We consider the synchronization problem of networked Euler-Lagrange systems with unknown parameters. The information flow in the network is represented by a directed communication graph and is subject to unknown and possibly discontinuous time-varying communication delays with unknown upper bounds. We propose a control scheme that achieves position synchronization, *i.e.*, all the positions of the systems converge to a common final position, provided that the directed communication graph contains a spanning tree. The convergence analysis of the proposed scheme is based on the multi-dimensional small-gain framework. Simulation results are presented that confirm the validity of the obtained results.

## I. INTRODUCTION

Recently, the synchronization problem of mechanical systems modeled by Euler-Lagrange dynamics has received a great attention in the research community, see for instance [1]-[4]. The main objective consists in designing a cooperative control scheme (using local information exchange) such that the individual systems reach an agreement on their states or on a common objective. In practical situations, the information exchange is generally restricted and is subject to communication delays.

Exploiting the passivity property of the systems, the authors in [5] proposed an output synchronization scheme with the assumption that the information exchange is represented by a balanced and strongly connected directed graph. Using a similar formulation, the cooperative trajectory tracking problem for multiple Euler-Lagrange systems has been addressed in [6]-[7]. More recently, the synchronization of nonlinear systems with relative degree two has been considered in [8]. The latter result can be applied to the class of Euler-Lagrange systems, however, the communication topology is assumed to be undirected. With the same assumption on the communication graph, a synchronization scheme that accounts for input saturations for networked Euler-Lagrange systems is proposed in [9]. In the case of general directed networks, the authors in [10] have shown that synchronization can be achieved with and without reference signals. In the aforementioned papers, it has been shown that synchronization is achieved despite the presence of delays inherent in communication systems, however, only the case of constant communication delays has been considered.

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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In practical situations involving networked systems, communication delays (that are unknown, time-varying and possibly discontinuous) must be seriously taken into consideration. The effects of time-varying communication delays are generally studied using Lyapunov-Krasovskii functionals to derive sufficient conditions on the communication delays so that synchronization is achieved. This can be seen in the wide literature related to teleoperation systems, see [11] for a survey. Also, some interesting results involving multiple nonlinear systems have been proposed in [12]-[13], for spacecraft formations, and in [14] for unmanned aerial vehicles. However, these results can be applied in the case of undirected networks, and require some assumptions on the communication delays, including their differentiability and/or known upper bounds.

The contribution of this paper consists in providing a solution to the synchronization problem of networked Euler-Lagrange systems in the presence of time-varying (possibly discontinuous) communication delays. The systems in the network are subject to parameter uncertainties and are interconnected according to a directed communication graph. The stability and convergence analysis presented in this work are based on the multi-dimensional small gain approach for systems with communication constraints; similar although not identical framework was previously developed in [15]. It is shown that the synchronization in the presence of unknown irregular communication delays is achieved under conditions that can be easily satisfied by an appropriate choice of the control gains. In particular, the approach doesn't impose any constraint on the upper bound of the communication delays. It should be pointed out that this work extends our earlier results in [16] to the case of systems with uncertain parameters and provides less restrictive conditions for synchronization. Simulation results on a network of ten robot manipulators are given to illustrate the performance of the proposed approach.

## II. PROBLEM STATEMENT

Consider a network of  $n$  not necessarily identical systems governed by the Euler-Lagrange equations of the form

$$\mathbf{M}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i + \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i + \mathbf{G}_i(\mathbf{q}_i) = \mathbf{u}_i, \quad (1)$$

for  $i \in \mathcal{N} \triangleq \{1, \dots, n\}$ , with  $\mathbf{q}_i \in \mathbb{R}^m$  is the vector of generalized configuration coordinates,  $\mathbf{M}_i(\mathbf{q}_i) \in \mathbb{R}^{m \times m}$  is the positive-definite inertia matrix,  $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i$  is the vector of Coriolis/Centrifugal forces,  $\mathbf{G}_i(\mathbf{q}_i)$  is the vector of gravitational force, and  $\mathbf{u}_i$  is the vector of torques associated

with the  $i^{\text{th}}$  system. We consider the following common properties of Euler-Lagrange systems:

- P.1 Each system in (1) admits a linear parametrization of the form  $\mathbf{M}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_{r_i} + \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_{r_i} + \mathbf{G}_i(\mathbf{q}_i) = \mathbf{Y}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i, \ddot{\mathbf{q}}_{r_i}, \ddot{\mathbf{q}}_{r_i})\boldsymbol{\theta}_i$ , with  $\mathbf{q}_{r_i} \in \mathbb{R}^m$  and  $\mathbf{Y}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i, \ddot{\mathbf{q}}_{r_i}, \ddot{\mathbf{q}}_{r_i}) \in \mathbb{R}^{m \times k}$  is a known regressor matrix and  $\boldsymbol{\theta}_i \in \mathbb{R}^k$  is the vector of the system's parameters.
- P.2 The Coriolis matrix  $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) \in \mathbb{R}^{m \times m}$  is defined such that  $\dot{\mathbf{M}}_i(\mathbf{q}_i) = \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) + \mathbf{C}_i^\top(\mathbf{q}_i, \dot{\mathbf{q}}_i)$ . Note that this property implies that the matrix  $\mathbf{M}_i(\mathbf{q}_i) - 2\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)$  is skew symmetric.
- P.3 For all  $\mathbf{q}_i, \mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ , there exists  $k_{c_i} \in \mathbb{R}_+$  such that  $\|\mathbf{C}_i(\mathbf{q}_i, \mathbf{x})\mathbf{y}\| \leq k_{c_i}\|\mathbf{x}\|\|\mathbf{y}\|$ . In addition,  $\mathbf{M}_i(\mathbf{q}_i)$  and  $\mathbf{G}_i(\mathbf{q}_i)$  are bounded independently from  $\mathbf{q}_i$ .

To achieve synchronization, the Euler-Lagrange systems exchange information over a network described by the directed interconnection graph  $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ . The set  $\mathcal{N}$  is the set of nodes or vertices, describing the set of systems in the network,  $\mathcal{E} \in \mathcal{N} \times \mathcal{N}$  is the set of ordered pairs of nodes, called edges, and  $\mathcal{A} = [k_{ij}] \in \mathbb{R}^{n \times n}$  is the weighted adjacency matrix. An edge  $(i, j)$  indicates that system  $j$  can receive information from system  $i$ , but not necessarily vice versa. The weighted adjacency matrix is defined such that  $k_{ii} \triangleq 0$ ,  $k_{ij} > 0$  if  $(j, i) \in \mathcal{E}$ , and  $k_{ij} = 0$  if  $(j, i) \notin \mathcal{E}$ . A directed path is a sequence of edges in a directed graph of the form  $(i_1, i_2), (i_2, i_3), \dots$ , where  $i_l \in \mathcal{N}$ . A directed graph is said to contain a directed spanning tree if there exists at least one node having a directed path to all of the other nodes. The Laplacian matrix  $\mathcal{L} := [l_{ij}] \in \mathbb{R}^{n \times n}$  of the directed graph  $\mathcal{G}$  is defined such that:  $l_{ii} = \sum_{j=1}^n k_{ij}$ , and  $l_{ij} = -k_{ij}$  for  $i \neq j$ . In view of its definition, the Laplacian matrix satisfies:  $\mathcal{L}\mathbf{1}_n = 0$ , with  $\mathbf{1}_n \in \mathbb{R}^n$  is the column vector of all ones. Moreover, if the directed graph has a directed spanning tree then  $\mathcal{L}$  has a single zero-eigenvalue and the rest of the spectrum of  $\mathcal{L}$  has positive real parts [17].

We assume that the model parameters of the systems in the network are not exactly known, each system can sense its state vector with no delay, and for any pair of nodes  $(j, i) \in \mathcal{E}$ , the information of  $j$ -th system is received by the  $i$ -th system with the communication delay  $\tau_{ij}(t)$ . The following assumption is imposed on the communication delays  $\tau_{ij}(t)$ .

*Assumption 1:* For each  $(j, i) \in \mathcal{E}$ , the communication delay  $\tau_{ij}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  can be decomposed into the sum of two terms,

$$\tau_{ij}(t) = \tau_{ij}^s(t) + \tau_{ij}^r(t), \quad (2)$$

where the components  $\tau_{ij}^s(\cdot)$  and  $\tau_{ij}^r(\cdot)$  have the following properties:

- i) There exists a function  $\tau^*: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\tau^*(t_2) - \tau^*(t_1) \leq t_2 - t_1$  for all  $t_1, t_2 \in \mathbb{R}_+$ , and  $|\tau_{ij}^s| \leq \tau^*(t)$  holds for all  $t \geq 0$ .
- ii) The function  $\tau_{ij}^s(t)$  satisfies:  $t - \tau_{ij}^s(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ .
- iii) There exists  $\Upsilon_{ij} \geq 0$  such that the inequality:  $|\tau_{ij}^s(t_2) - \tau_{ij}^s(t_1)| \leq \Upsilon_{ij} \cdot |t_2 - t_1|$  holds for almost all  $t_2, t_1 \in \mathbb{R}_+$ , with  $t_2 \geq t_1$ .

- iv) There exists  $\Delta_{ij}^r \geq 0$  such that:  $|\tau_{ij}^r(t)| \leq \Delta_{ij}^r$  holds for almost all  $t \geq 0$ .

The subscripts  $s$  and  $r$  indicate that  $\tau_{ij}^s(\cdot)$  and  $\tau_{ij}^r(\cdot)$  are the ‘‘smooth’’ and the ‘‘irregular’’ components of the communication delay, respectively. In particular, part i) implies the existence of an upper bound of the smooth part of the communication delays, given by  $\tau^*$ , which is possibly a time-varying unbounded function that does not grow faster than the time itself. Also, part iii) implies that the time derivative  $d\tau_{ij}^s(t)/dt$  is well-defined for almost all  $t \geq 0$  and satisfies  $|d\tau_{ij}^s(t)/dt| \leq \Upsilon_{ij}$ , where defined.

Our objective is to design a control scheme that achieves synchronization such that all systems synchronize their positions, *i.e.*,  $(\mathbf{q}_i - \mathbf{q}_j) \rightarrow 0$ , for  $i, j \in \mathcal{N}$ , with  $\dot{\mathbf{q}}_i \rightarrow 0$ , for  $i \in \mathcal{N}$  as  $t \rightarrow +\infty$ .

### III. PRELIMINARY RESULTS

In this section, we present some definitions and technical results that will be used in the subsequent analysis. Consider an affine nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + \dots + g_p(x)u_p, \\ y_1 &= h_1(x), \\ &\vdots \\ y_q &= h_q(x), \end{aligned} \quad (3)$$

where  $x \in \mathbb{R}^N$ ,  $u_i \in \mathbb{R}^{\tilde{m}_i}$  for  $i \in \mathcal{N}_p := \{1, \dots, p\}$ ,  $y_j \in \mathbb{R}^{\tilde{m}_j}$  for  $j \in \mathcal{N}_q := \{1, \dots, q\}$ , and  $f(\cdot)$ ,  $g_i(\cdot)$ , for  $i \in \mathcal{N}_p$ , and  $h_j(\cdot)$ , for  $j \in \mathcal{N}_q$ , are locally Lipschitz functions of the corresponding dimensions,  $f(0) = 0$ ,  $h(0) = 0$ . We assume that for any initial condition  $x(t_0)$  and any inputs  $u_1(t), \dots, u_p(t)$  that are uniformly essentially bounded on  $[t_0, t_1]$ , the corresponding solution  $x(t)$  is well defined for all  $t \in [t_0, t_1]$ .

*Definition 1:* A system of the form (3) is said to be *weakly input-to-output stable* (WIOS) if there exist  $^1 \beta_i \in \mathcal{K}_\infty$  and  $\gamma_{ij} \in \mathcal{K}$ , with  $i \in \mathcal{N}_q$  and  $j \in \mathcal{N}_p$ , such that the following inequalities hold along the trajectories of the system for any uniformly bounded inputs and for all  $i \in \mathcal{N}_q$ :

- i) *uniform boundedness:*

$$\begin{aligned} |y_i(t)| \leq & \beta_i(|x(t_0)|) + \gamma_{i1} \left( \sup_{s \in [t_0, t]} |u_1(s)| \right) + \dots \\ & + \gamma_{ip} \left( \sup_{s \in [t_0, t]} |u_p(s)| \right), \quad \forall t_0, t \in \mathbb{R}, t \geq t_0, \end{aligned}$$

- ii) *asymptotic gain:*

$$\begin{aligned} \limsup_{t \rightarrow +\infty} |y_i(t)| \leq & \gamma_{i1} \left( \limsup_{t \rightarrow +\infty} |u_1(t)| \right) + \dots \\ & + \gamma_{ip} \left( \limsup_{t \rightarrow +\infty} |u_p(t)| \right). \end{aligned}$$

In the definition above, the function  $\gamma_{ij} \in \mathcal{K}$ , where  $i \in \mathcal{N}_q$ ,  $j \in \mathcal{N}_p$ , is called the IOS (input-to-output stability) gain from the input  $u_j$  to the output  $y_i$ . In the sequel analysis, we will mostly deal with the case where the IOS gains are linear

<sup>1</sup> A continuous function  $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathcal{K}$  ( $\gamma \in \mathcal{K}$ ) if it is strictly increasing and satisfies  $\gamma(0) = 0$ . A function  $\gamma \in \mathcal{K}$  belongs to class  $\mathcal{K}_\infty$  if  $\gamma(s) \rightarrow \infty$  as  $s \rightarrow \infty$  [18].

functions of the form  $\gamma_{ij}(s) := \gamma_{ij}^0 \cdot s$  for each  $s \geq 0$ , where  $\gamma_{ij}^0 \geq 0$ . In this case, we will simply say that the system has a linear IOS gain  $\gamma_{ij}^0 \geq 0$ . It can be checked directly that for system (3), the input-to-state stability (ISS) [19] implies the WIOS property defined above. Also, the input-to-output stability (IOS) property [19] implies WIOS, but the converse does not hold as demonstrated in [19, p. 194].

The following small-gain theorem is the key technical tool used in our work.

*Theorem 1:* Consider a system of the form (3). Suppose the system is WIOS, and the corresponding linear IOS gains  $\gamma_{ij}^0 \geq 0$ . Suppose also that each input  $u_j(\cdot)$ ,  $j \in \mathcal{N}_p$ , is a Lebesgue measurable function satisfying

$$u_j(t) \equiv 0 \quad \text{for } t < 0, \quad (4)$$

$$|u_j(t)| \leq \sum_{i \in \mathcal{N}_q} \mu_{ji} \cdot \left( \sup_{s \in [t - \tau_{ji}(t), t]} |y_i(s)| \right), \quad (5)$$

for almost all  $t \geq 0$  where  $\mu_{ji} \geq 0$ , and  $\tau_{ji}(t)$  satisfy Assumption 1. Let  $\Gamma := \Gamma^0 \cdot \mathcal{M} \in \mathbb{R}^{q \times q}$ , where  $\Gamma^0 := \{\gamma_{ij}^0\}$ ,  $\mathcal{M} := \{\mu_{ji}\}$ ,  $i \in \mathcal{N}_q$ ,  $j \in \mathcal{N}_p$ . If

$$\rho(\Gamma) < 1, \quad (6)$$

where  $\rho(\Gamma)$  is the spectral radius of the matrix  $\Gamma$ , then the trajectories of the system (3) with input-output constraints (4), (5) are well defined for all  $t \geq 0$  and such that all the outputs  $y_i(t)$ ,  $i \in \mathcal{N}_q$ , and all the inputs  $u_j(\cdot)$ ,  $j \in \mathcal{N}_p$ , are uniformly bounded and satisfy  $|y_i(t)| \rightarrow 0$ ,  $|u_j(t)| \rightarrow 0$  as  $t \rightarrow +\infty$ .

*Proof:* The proof is omitted due to space limitations. ■

#### IV. MAIN RESULT

We propose the following control algorithm for each system in (1)

$$\begin{aligned} \mathbf{u}_i &= \mathbf{Y}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i, \boldsymbol{\eta}_i, \dot{\boldsymbol{\eta}}_i) \hat{\boldsymbol{\theta}}_i - k_i^s (\dot{\mathbf{q}}_i - \boldsymbol{\eta}_i), \\ \dot{\hat{\boldsymbol{\theta}}}_i &= -\Pi_i \mathbf{Y}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i, \boldsymbol{\eta}_i, \dot{\boldsymbol{\eta}}_i)^\top (\dot{\mathbf{q}}_i - \boldsymbol{\eta}_i), \end{aligned} \quad (7)$$

where  $\mathbf{Y}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i, \boldsymbol{\eta}_i, \dot{\boldsymbol{\eta}}_i) \hat{\boldsymbol{\theta}}_i = \hat{\mathbf{M}}_i(\mathbf{q}_i) \dot{\boldsymbol{\eta}}_i + \hat{\mathbf{C}}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) \boldsymbol{\eta}_i + \hat{\mathbf{G}}_i(\mathbf{q}_i)$ , with  $\hat{\mathbf{M}}_i$ ,  $\hat{\mathbf{C}}_i$  and  $\hat{\mathbf{G}}_i$  being, respectively, known estimates of  $\mathbf{M}_i$ ,  $\mathbf{C}_i$  and  $\mathbf{G}_i$ . The vector  $\hat{\boldsymbol{\theta}}_i \in \mathbb{R}^k$  is the estimate of  $\boldsymbol{\theta}_i$  defined in property P.1, with  $\hat{\boldsymbol{\theta}}_i(0)$  can be selected arbitrarily. The matrix  $\Pi_i$  is symmetric positive definite and  $k_i^s > 0$  is a scalar gain. The vector  $\boldsymbol{\eta}_i$  is the solution of the following dynamic system

$$\dot{\boldsymbol{\eta}}_i = -k_i^\eta \boldsymbol{\eta}_i - \lambda_i \left( \kappa_i \mathbf{q}_i - \psi_i^{\{1\}} \right), \quad (8)$$

where  $\boldsymbol{\eta}_i(0)$  can be selected arbitrarily,  $\kappa_i := (\sum_{j=1}^n k_{ij})$ ,  $k_{ij}$  is the  $(i, j)$ -th element of the adjacency matrix  $\mathcal{A}$  of the directed communication graph  $\mathcal{G}$ ,  $k_i^\eta$ , and  $\lambda_i$ , are strictly positive scalar gains, and  $\psi_i^{\{1\}}$  is the output of the following second-order filter:

$$\begin{cases} \psi_i^{\{1\}} &= \psi_i^{\{2\}} \\ \psi_i^{\{2\}} &= -\alpha_1 \psi_i^{\{2\}} - \alpha_0 \psi_i^{\{1\}} \\ &+ \alpha_0 \sum_{j=1}^n k_{ij} \mathbf{q}_j(t - \tau_{ij}(t)) \end{cases}, \quad (9)$$

where  $\psi_i^{\{1\}}(0)$ ,  $\psi_i^{\{2\}}(0)$  can be selected arbitrarily,  $\alpha_1$ , and  $\alpha_0$  are strictly positive scalar gains.

Now, denote

$$\begin{aligned} \nu &:= -\max\{\mathcal{R}e(\nu_1), \mathcal{R}e(\nu_2)\}, \\ \mu_i &:= -\max(\mathcal{R}e(\mu_{i,1}), \mathcal{R}e(\mu_{i,2})), \end{aligned} \quad (10)$$

where  $\nu_1, \nu_2$  are the roots of  $p^2 + \alpha_1 p + \alpha_0 = 0$ , and  $\mu_{i,1}, \mu_{i,2}$  are the roots of  $p^2 + k_i^\eta p + \lambda_i \kappa_i = 0$ .

Our main result is the following theorem.

*Theorem 2:* Consider the network of  $n$ -systems described by (1), where the interconnection between the systems is described by the directed communication graph  $\mathcal{G}$ . Let the controller be defined by (7)-(9) and suppose Assumption 1 holds. If the control gains of each  $i$ -th system with  $\kappa_i \neq 0$  satisfy<sup>2</sup>

$$\mu_i \cdot \nu > \sum_{j=1}^n k_{ij} (1 + \Upsilon_{ij} + \alpha_0 \cdot \Delta_{ij}^\tau), \quad (11)$$

then the trajectories of the closed-loop system (1), (7)-(9) are uniformly bounded and  $\dot{\mathbf{q}}_i \rightarrow 0$ ,  $\tilde{\mathbf{q}}_i \rightarrow 0$ , and  $\sum_{j=1}^n k_{ij} (\mathbf{q}_i - \mathbf{q}_j(t - \tau_{ij}(t))) \rightarrow 0$ , as  $t \rightarrow +\infty$ , for all  $i \in \mathcal{N}$ . Furthermore, if the directed communication graph  $\mathcal{G}$  contains a directed spanning tree, and  $\tau^*(t)$  in Assumption 1, point i), satisfies  $\limsup_{t \rightarrow +\infty} \tau^*(t) < \infty$ , then all systems synchronize their positions to the same final position, *i.e.*,  $\mathbf{q}_i \rightarrow \mathbf{q}_c$ , for  $i \in \mathcal{N}$  and some vector  $\mathbf{q}_c \in \mathbb{R}^n$ .

*Remark 1:* Note that the small-gain condition (11) does not impose additional constraints on the communication delays, and can be easily satisfied with an appropriate choice of the control gains if Assumption 1 holds.

#### V. PROOF OF THEOREM 2

Let define the error variable  $\mathbf{s}_i := (\dot{\mathbf{q}}_i - \boldsymbol{\eta}_i)$ , which is governed in view of (1) and (7) as

$$\begin{aligned} \mathbf{M}_i \dot{\mathbf{s}}_i + \mathbf{C}_i \mathbf{s}_i + k_i^s \mathbf{s}_i &= \mathbf{Y}_i \tilde{\boldsymbol{\theta}}_i, \\ \dot{\tilde{\boldsymbol{\theta}}}_i &= -\Pi_i \mathbf{Y}_i^\top \mathbf{s}_i, \end{aligned} \quad (12)$$

where the arguments of  $\mathbf{M}_i$ ,  $\mathbf{C}_i$  and  $\mathbf{Y}_i$  have been omitted for simplicity,  $\tilde{\boldsymbol{\theta}}_i = (\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i)$ , and we have used Property P.1. Also, for our purposes, it is convenient to introduce the following variables,

$$\begin{aligned} \tilde{\mathbf{q}}_i &:= \kappa_i \mathbf{q}_i - \psi_i^{\{1\}}, \\ \hat{\mathbf{q}}_i(t) &:= \sum_{j=1}^n k_{ij} \mathbf{q}_j(t - \tau_{ij}(t)), \\ \hat{\mathbf{q}}_i^s(t) &:= \sum_{j=1}^n k_{ij} \mathbf{q}_j(t - \tau_{ij}^s(t)), \\ \Delta \hat{\mathbf{q}}_i(t) &:= \hat{\mathbf{q}}_i(t) - \hat{\mathbf{q}}_i^s(t), \\ \tilde{\psi}_i &:= \psi_i^{\{1\}} - \hat{\mathbf{q}}_i^s, \end{aligned} \quad (13)$$

with  $\tau_{ij}^s(t)$  given in Assumption 1. First, we consider all agents  $i \in \mathcal{N}$  with  $\kappa_i = \sum_{j=1}^n k_{ij} \neq 0$ . Using (12) and (8)-(9) along with the definition of  $\mathbf{s}_i$  and (13), the closed loop

<sup>2</sup>Note that  $\kappa_i = 0$  indicates that the  $i$ -th system does not receive information from any other system in the network. Therefore, condition (11) is imposed only on systems that receive information from at least one other neighbor in the team.

dynamics of the  $i$ -th system with  $\kappa_i \neq 0$  is obtained as

$$\dot{\mathbf{s}}_i = \mathbf{M}_i^{-1} \left( \mathbf{Y}_i \tilde{\boldsymbol{\theta}}_i - \mathbf{C}_i \mathbf{s}_i - k_d^s \mathbf{s}_i \right) \quad (14)$$

$$\dot{\tilde{\boldsymbol{\theta}}}_i = -\Pi_i \mathbf{Y}_i^\top \mathbf{s}_i \quad (15)$$

$$\dot{\hat{\mathbf{q}}}_i = \kappa_i \boldsymbol{\eta}_i + \kappa_i \mathbf{s}_i - \psi_i^{\{2\}} \quad (16)$$

$$\dot{\boldsymbol{\eta}}_i = -k_i^\eta \boldsymbol{\eta}_i - \lambda_i \tilde{\mathbf{q}}_i \quad (17)$$

$$\dot{\tilde{\psi}}_i = \psi_i^{\{2\}} - \hat{\mathbf{q}}_i^s \quad (18)$$

$$\dot{\psi}_i^{\{2\}} = -\alpha_0 \tilde{\psi}_i - \alpha_1 \psi_i^{\{2\}} + \alpha_0 \Delta \hat{\mathbf{q}}_i \quad (19)$$

where the vectors  $\mathbf{s}_i$ ,  $\tilde{\psi}_i$ ,  $\psi_i^{\{2\}}$ ,  $\boldsymbol{\eta}_i$  and  $\tilde{\mathbf{q}}_i$  are the states,  $\hat{\mathbf{q}}_i^s$  and  $\Delta \hat{\mathbf{q}}_i$  are the inputs, and the output is given by

$$\dot{\mathbf{q}}_i = \mathbf{s}_i + \boldsymbol{\eta}_i. \quad (20)$$

*Proposition 1:* The system (14)-(19) with inputs  $\hat{\mathbf{q}}_i^s$ ,  $\Delta \hat{\mathbf{q}}_i$  and output (20) is weakly IOS. Moreover, the IOS gains with respect to inputs  $\hat{\mathbf{q}}_i^s$ ,  $\Delta \hat{\mathbf{q}}_i$  are  $\frac{1}{\mu_i \nu}$  and  $\frac{\alpha_0}{\mu_i \nu}$ , respectively.

*Proof:* Applying Lemma 1, given in Appendix A, to the trajectories of the system (18)-(19), we see that the following estimate

$$\begin{aligned} \left| \begin{array}{l} \tilde{\psi}_i(t) \\ \psi_i^{\{2\}}(t) \end{array} \right| &\leq e^{-\nu(t-t_0)} \left| \begin{array}{l} \tilde{\psi}_i(t_0) \\ \psi_i^{\{2\}}(t_0) \end{array} \right| + \frac{1}{\nu} \sup_{\sigma \in [t_0, t]} \left| \hat{\mathbf{q}}_i^s(\sigma) \right| \\ &+ \frac{\alpha_0}{\nu} \sup_{\sigma \in [t_0, t]} |\Delta \hat{\mathbf{q}}_i(\sigma)|, \end{aligned} \quad (21)$$

holds for any  $t \geq t_0$ , where  $\nu > 0$  is defined by (10). Inequality (21) indicates, in particular that the system (18)-(19) is ISS with respect to inputs  $\hat{\mathbf{q}}_i^s$ ,  $\Delta \hat{\mathbf{q}}_i$ , with ISS gains  $\frac{1}{\nu}$  and  $\frac{\alpha_0}{\nu}$ , respectively. Similarly, Applying Lemma 1 to (16)-(17) yields us to write

$$\begin{aligned} \left| \begin{array}{l} \tilde{\mathbf{q}}_i(t) \\ \boldsymbol{\eta}_i(t) \end{array} \right| &\leq e^{-\mu_i(t-t_0)} \left| \begin{array}{l} \tilde{\mathbf{q}}_i(t_0) \\ \boldsymbol{\eta}_i(t_0) \end{array} \right| + \frac{\kappa_i}{\mu_i} \sup_{\sigma \in [t_0, t]} |\mathbf{s}_i(\sigma)| \\ &+ \frac{1}{\mu_i} \sup_{\sigma \in [t_0, t]} \left| \psi_i^{\{2\}}(\sigma) \right|, \end{aligned} \quad (22)$$

which implies that the system (16)-(17) is ISS with respect to inputs  $\mathbf{s}_i$ ,  $\psi_i^{\{2\}}$ . In particular,  $1/\mu_i$  is the ISS gain of system (16)-(17) with respect to the input  $\psi_i^{\{2\}}$ . Taking into account that  $\psi_i^{\{2\}}$  is a part of the state of (18)-(19), and the fact that a cascade connection of two ISS subsystems is ISS [19], one concludes that the system (16)-(19) is ISS with respect to inputs  $\hat{\mathbf{q}}_i^s$ ,  $\Delta \hat{\mathbf{q}}_i$ ,  $\mathbf{s}_i$ . On the other hand, the uniform boundedness of  $\mathbf{s}_i$ ,  $\tilde{\boldsymbol{\theta}}_i$  (with the upper bound independent on other variables) can be shown using the following Lyapunov function candidate  $V = \frac{1}{2} \sum_{i=1}^n \left( \mathbf{s}_i^\top \mathbf{M}_i(\mathbf{q}_i) \mathbf{s}_i + \tilde{\boldsymbol{\theta}}_i^\top \Pi_i^{-1} \tilde{\boldsymbol{\theta}}_i \right)$ , whose time-derivative along the trajectories of (14)-(15) is  $\dot{V} = -\sum_{i=1}^n k_i^d \mathbf{s}_i^\top \mathbf{s}_i$ . Combination of these two facts proves the uniform boundedness property given in Definition 1.

To prove the asymptotic gain property, note that the input-to-state stability of (16)-(19) implies that  $\boldsymbol{\eta}_i$  and  $\dot{\boldsymbol{\eta}}_i$  are uniformly bounded for any uniformly bounded inputs  $\hat{\mathbf{q}}_i^s$ ,  $\Delta \hat{\mathbf{q}}_i$ , since  $\mathbf{s}_i$  was shown to be uniformly bounded. Consequently, we can see, using properties P1 and P3, that the right-hand side of (14) is uniformly bounded, and hence  $\dot{\mathbf{s}}_i$  is uniformly bounded. This implies that  $\dot{V} = -2 \cdot \sum_{i=1}^n k_i^d \mathbf{s}_i^\top \mathbf{s}_i$

is uniformly bounded, and therefore the uniform continuity of  $\dot{V}(t)$ . Now, applying Barbälát lemma [18, Lemma 8.2], we conclude that  $\dot{V}(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , and therefore  $\mathbf{s}_i \rightarrow 0$  as  $t \rightarrow +\infty$ . This, with the ISS property of (16)-(19), lead us to conclude that the system (14)-(19), (20) is weakly IOS. The estimates of IOS gains can be verified directly by combining (21), (22), and (20). The proof of the proposition is complete. ■

Now, let us consider the  $i$ -th agents with  $\kappa_i = 0$ . In this case,  $\hat{\mathbf{q}}_i = -\psi_i^{\{1\}} = -\tilde{\psi}_i$  and the corresponding closed loop dynamics becomes

$$\dot{\mathbf{s}}_i = \mathbf{M}_i^{-1} \left( \mathbf{Y}_i \tilde{\boldsymbol{\theta}}_i - \mathbf{C}_i \mathbf{s}_i - k_d^s \mathbf{s}_i \right), \quad (23)$$

$$\dot{\tilde{\boldsymbol{\theta}}}_i = -\Pi_i \mathbf{Y}_i^\top \mathbf{s}_i, \quad (24)$$

$$\dot{\boldsymbol{\eta}}_i = -k_i^\eta \boldsymbol{\eta}_i + \lambda_i \tilde{\psi}_i \quad (25)$$

$$\dot{\tilde{\psi}}_i = \psi_i^{\{2\}}, \quad (26)$$

$$\dot{\psi}_i^{\{2\}} = -\alpha_0 \tilde{\psi}_i - \alpha_1 \psi_i^{\{2\}}, \quad (27)$$

with output  $\dot{\mathbf{q}}_i = \mathbf{s}_i + \boldsymbol{\eta}_i$ . Using the same Lyapunov function in the proof of Proposition 1, and by noting that system (25)-(27) is exponentially stable, we can conclude that (23)-(27) has uniformly bounded state trajectories, and  $\dot{\mathbf{q}}_i \rightarrow 0$  as  $t \rightarrow +\infty$ . Therefore, one can formally consider (23)-(27) to be similar to (14)-(19) with inputs  $\hat{\mathbf{q}}_i^s$  and  $\Delta \hat{\mathbf{q}}_i$ , with respect to which the system is WIOS with zero IOS gains.

From the above analysis, the networked systems (1) with input (7)-(9) can be considered as a system that consists of  $n$  subsystems of the form (14)-(19), where  $i \in \mathcal{N}$ . Each  $i$ -th subsystem (14)-(19) has two inputs,  $\hat{\mathbf{q}}_i^s$ ,  $\Delta \hat{\mathbf{q}}_i$ , and one output  $\dot{\mathbf{q}}_i$ . Therefore, the overall system has  $2n$  inputs and  $n$  outputs. For our purposes, it is convenient to order them as follows:  $y_k := \dot{\mathbf{q}}_k$  for  $k \in \mathcal{N}$ , and  $u_{2k-1} := \hat{\mathbf{q}}_k^s$ ,  $u_{2k} := \Delta \hat{\mathbf{q}}_k$  for  $k \in \mathcal{N}$ . Proposition 1 indicates that thus defined system is weakly IOS. Moreover, based on the above described order of the inputs and outputs, the elements of the IOS gain matrix  $\Gamma^0 := \{\gamma_{ij}^0\}$  are as follows

$$\gamma_{il}^0 = \begin{cases} \frac{1}{\mu_i \nu} & \text{if } l = 2i - 1, i \in \mathcal{N}, \kappa_i \neq 0, \\ \frac{\alpha_0}{\mu_i \nu} & \text{if } l = 2i, i \in \mathcal{N}, \kappa_i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, using Assumption 1, the following estimates of the inputs  $|\Delta \hat{\mathbf{q}}_i|$  and  $|\hat{\mathbf{q}}_i^s|$  can be derived

$$\begin{aligned} |\Delta \hat{\mathbf{q}}_i(t)| &= \left| \sum_{j=1}^n k_{ij} [\mathbf{q}_j(t - \tau_{ij}(t)) - \mathbf{q}_j(t - \tau_{ij}^s(t))] \right| \\ &\leq \sum_{j=1}^n k_{ij} |\mathbf{q}_j(t - \tau_{ij}(t)) - \mathbf{q}_j(t - \tau_{ij}^s(t))| \\ &\leq \sum_{j=1}^n k_{ij} \cdot \Delta \tau_{ij}^\tau \cdot \left( \sup_{\sigma \in [t_1, t_2]} |\dot{\mathbf{q}}_j(\sigma)| \right), \end{aligned} \quad (28)$$

$$\begin{aligned} \left| \dot{\hat{\mathbf{q}}}_i^s(t) \right| &= \left| \sum_{j=1}^n k_{ij} \dot{\mathbf{q}}_j(t - \tau_{ij}^s(t)) \left[ 1 - \frac{d\tau_{ij}^s(t)}{dt} \right] \right| \\ &\leq \sum_{j=1}^n k_{ij} (1 + \Upsilon_{ij}) \left| \dot{\mathbf{q}}_j(t - \tau_{ij}^s(t)) \right|, \end{aligned} \quad (29)$$

where, in the latter inequality, we use notation  $t_1 := (t - \max\{\tau_{ij}(t), \tau_{ij}^s(t)\})$  and  $t_2 := (t - \min\{\tau_{ij}(t), \tau_{ij}^s(t)\})$ . Therefore, the elements of the interconnection matrix  $\mathcal{M} := \{\mu_{ji}\}$  are as follows:

$$\mu_{li} = \begin{cases} k_{ij} (1 + \Upsilon_{ij}) & \text{if } l = 2j - 1, j \in \mathcal{N}, i \in \mathcal{N} \\ k_{ij} \cdot \Delta_{ij}^\tau & \text{if } l = 2j, j \in \mathcal{N}, i \in \mathcal{N}. \end{cases}$$

From the above expressions, we conclude that the elements of the closed-loop gain matrix  $\Gamma := \Gamma^0 \cdot \mathcal{M} = \{\gamma_{ij}\}_{i,j \in \mathcal{N}}$  are as follows

$$\gamma_{ij} = \begin{cases} \frac{k_{ij}}{\mu_i \cdot \nu} (1 + \Upsilon_{ij} + \alpha_0 \cdot \Delta_{ij}^\tau), & \text{if } \kappa_i \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

Moreover, notice that  $k_{ii} = 0$  for all  $i \in \mathcal{N}$ , which implies  $\gamma_{ii} = 0$  for all  $i \in \mathcal{N}$ , *i.e.*, the diagonal elements of  $\Gamma$  are all zeros. Taking into account that the elements of  $\Gamma$  are nonnegative, one can apply the Geršgorin disc theorem [20] to conclude that  $\rho(\Gamma) < 1$  if

$$\Gamma_i := \frac{1}{\mu_i \cdot \nu} \sum_{j=1}^n k_{ij} (1 + \Upsilon_{ij} + \alpha_0 \cdot \Delta_{ij}^\tau) < 1,$$

which is satisfied by (11). Now, applying the small gain theorem (Theorem 1), we conclude that all  $\dot{\mathbf{q}}_i(t)$ ,  $\dot{\hat{\mathbf{q}}}_i^s(t)$  and  $\Delta \hat{\mathbf{q}}_i(t)$ , for  $i \in \mathcal{N}$ , are uniformly bounded and  $|\dot{\mathbf{q}}_i(t)| \rightarrow 0$ ,  $|\dot{\hat{\mathbf{q}}}_i^s(t)| \rightarrow 0$ ,  $|\Delta \hat{\mathbf{q}}_i(t)| \rightarrow 0$  as  $t \rightarrow +\infty$ . This, with Proposition 1, lead to the conclusion that  $\tilde{\mathbf{q}}_i$ ,  $\boldsymbol{\eta}_i$ ,  $\tilde{\psi}_i$ ,  $\psi_i^{\{2\}}$  are uniformly bounded and  $\tilde{\mathbf{q}}_i \rightarrow 0$ ,  $\boldsymbol{\eta}_i \rightarrow 0$ ,  $\tilde{\psi}_i \rightarrow 0$ ,  $\psi_i^{\{2\}} \rightarrow 0$  as  $t \rightarrow +\infty$ . Then, using (13) with  $|\Delta \hat{\mathbf{q}}_i(t)| \rightarrow 0$ , we have  $(\psi_i^{\{1\}} - \hat{\mathbf{q}}_i^s) \rightarrow (\psi_i^{\{1\}} - \hat{\mathbf{q}}_i) \rightarrow 0$ , which implies from (13) that  $\sum_{j=1}^n k_{ij} (\mathbf{q}_i - \mathbf{q}_j(t - \tau_{ij}(t))) \rightarrow 0$  as  $t \rightarrow +\infty$ , for  $i \in \mathcal{N}$ .

To prove the last point in the theorem, note that we can verify from parts i) and iv) of Assumption 1, with  $\limsup_{t \rightarrow +\infty} \tau^*(t) < \infty$  and  $\dot{\mathbf{q}}_i \rightarrow 0$  as  $t \rightarrow +\infty$ , that  $(\mathbf{q}_j - \mathbf{q}_j(t - \tau_{ij}(t))) := \int_{t-\tau_{ij}(t)}^t \dot{\mathbf{q}}_j(s) ds \rightarrow 0$  as  $t \rightarrow +\infty$  for all  $i, j \in \mathcal{N}$ . Then, exploiting the above results with the relation  $(\mathbf{q}_i - \mathbf{q}_j(t - \tau_{ij}(t))) = (\mathbf{q}_i - \mathbf{q}_j + \int_{t-\tau_{ij}(t)}^t \dot{\mathbf{q}}_j(s) ds)$ , we conclude that  $\sum_{j=1}^n k_{ij} (\mathbf{q}_i - \mathbf{q}_j) \rightarrow 0$ , for  $i \in \mathcal{N}$ , which is equivalent to  $(\mathcal{L} \otimes \mathbf{I}_m) \mathbf{Q} \rightarrow 0$  where  $\mathcal{L}$  is the Laplacian matrix of the communication graph  $\mathcal{G}$ ,  $\mathbf{Q} \in \mathbb{R}^{nm}$  is the vector containing all  $\mathbf{q}_i$  for  $i \in \mathcal{N}$ , and  $\otimes$  is the Kronecker product. With the condition that the communication graph contains a spanning tree, we conclude following similar arguments as in [17] that the only solution to  $(\mathcal{L} \otimes \mathbf{I}_m) \mathbf{Q} \rightarrow 0$  is  $\mathbf{Q} \rightarrow (\mathbf{1}_n \otimes \mathbf{q}_c)$ , for some  $\mathbf{q}_c \in \mathbb{R}^m$ . As a result, we conclude that  $\mathbf{q}_i \rightarrow \mathbf{q}_c$  for  $i \in \mathcal{N}$ .

## VI. SIMULATION RESULTS

In this section, we provide simulation results of a network of ten planar two degrees of freedom rigid manipulator arms (with revolute joints). For any generalized coordinates  $\mathbf{q}_i = (q_{i1}, q_{i2})^\top \in \mathbb{R}^2$ , the inertia matrices, Coriolis and centrifugal matrices, and gravity vectors are given, respectively, by:  $\mathbf{M}_i(\mathbf{q}_i) = [m_{ij}]_{2 \times 2}$ ,  $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) = [c_{ij}]_{2 \times 2}$ , and  $\mathbf{G}_i(\mathbf{q}_i) = [G_{i1}, G_{i2}]^\top$  with  $m_{i11} = \theta_1 + 2\theta_2 \cos(q_{i2})$ ,  $m_{i12} = m_{i21} = \theta_3 + \theta_2 \cos(q_{i2})$ ,  $m_{i22} = \theta_3$ ,  $c_{i11} = -\theta_2 \sin(q_{i2}) \dot{q}_{i2}$ ,  $c_{i12} = -\theta_2 \sin(q_{i2}) (\dot{q}_{i1} + \dot{q}_{i2})$ ,  $c_{i21} = \theta_2 \sin(q_{i2}) \dot{q}_{i1}$ ,  $c_{i22} = 0$ ,  $G_{i1} = g\theta_5 \cos(q_{i1}) + g\theta_4 \cos(q_{i1} + q_{i2})$ , and  $G_{i2} = g\theta_4 \cos(q_{i1} + q_{i2})$ . In these expressions,  $g = 9.81 \text{ m/sec}^2$ , the variables  $\theta_k$ ,  $k = 1, \dots, 5$ , are given as:  $\theta_1 = (m_1 l_{c1}^2 + m_2 (l_1^2 + l_2^2)) + I_1 + I_2$ ,  $\theta_2 = m_2 l_1 l_{c2}$ ,  $\theta_3 = (m_2 l_{c2}^2 + I_2)$ ,  $\theta_4 = m_2 l_{c2}$ , and  $\theta_5 = (m_1 l_{c1} + m_2 l_1)$ , with  $m_1 = m_2 = 1 \text{ kg}$ ,  $l_1 = l_2 = 0.5 \text{ m}$ ,  $l_{c1} = l_{c2} = 0.25 \text{ m}$ , and  $I_1 = I_2 = 0.1 \text{ kg/m}^2$ . The following parametrization is considered for the control law (7):  $\mathbf{Y}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i, \eta_i, \dot{\eta}_i) = [Y_{ijk}] \in \mathbb{R}^{2 \times 5}$ , with  $Y_{i11} = \dot{\eta}_{i1}$ ,  $Y_{i12} = \cos(q_{i2})(2\dot{\eta}_{i1} + \dot{\eta}_{i2}) - \sin(q_{i2})(\eta_{i1} \dot{q}_{i2} + \eta_{i2}(\dot{q}_{i1} + \dot{q}_{i2}))$ ,  $Y_{i13} = \dot{\eta}_{i2}$ ,  $Y_{i14} = Y_{i24} = g \cos(q_{i1} + q_{i2})$ ,  $Y_{i15} = g \cos(q_{i1})$ ,  $Y_{i21} = Y_{i25} = 0$ ,  $Y_{i22} = \dot{\eta}_{i1} \cos(q_{i2}) + \eta_{i1} \dot{q}_{i1} \sin(q_{i2})$ ,  $Y_{i23} = \dot{\eta}_{i1} + \dot{\eta}_{i2}$ , and  $\eta_i := (\eta_{i1}, \eta_{i2})^\top = -\Lambda_i \tilde{\mathbf{q}}_i$ . The vector of estimated parameters is  $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)^\top$ .

We implement the control scheme in Theorem 2, with  $\hat{\boldsymbol{\theta}}(0) = (1, 1, 1, 1, 1)^\top$ ,  $\boldsymbol{\eta}_i(0) = \psi_i^{\{1\}}(0) = \psi_i^{\{2\}}(0) = 0$ , for  $i \in \mathcal{N}$ , and the communication topology between the systems is represented by the directed graph  $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{K})$ , with  $\mathcal{N} = \{1, \dots, 10\}$ ,  $\mathcal{E} = \{(1, 2), (1, 10), (2, 3), (4, 1), (4, 5), (5, 6), (6, 7), (7, 8), (8, 7), (8, 9), (9, 1), (9, 3), (9, 10), (10, 8)\}$ , and the adjacency matrix, and the adjacency matrix  $\mathcal{A} = [k_{ij}]$ , with  $k_{ij} = 0.2$ , for  $(j, i) \in \mathcal{E}$ , and zero otherwise. It can be easily verified that  $\mathcal{G}$  contains a spanning tree. We consider the following communication delays for each  $(j, i) \in \mathcal{E}$ :  $\tau_{ij}(t) = \bar{\tau}_{ij} \phi(t)$  sec, with  $\phi(t) = (1 - \cos(0.25t + 1) + 0.25r(t))$  sec,  $r(t) \in [0, 1]$  is a uniform random function, and the constants  $\bar{\tau}_{1i} = 0.6$ ,  $\bar{\tau}_{2i} = 0.6$ ,  $\bar{\tau}_{3i} = 0.6$ ,  $\bar{\tau}_{4i} = 0.6$ ,  $\bar{\tau}_{5i} = 0.5$ ,  $\bar{\tau}_{6i} = 0.5$ ,  $\bar{\tau}_{7i} = 0.6$ ,  $\bar{\tau}_{8i} = 0.5$ ,  $\bar{\tau}_{9i} = 0.6$ ,  $\bar{\tau}_{10i} = 0.6$ , for  $i \in \mathcal{N}$ . It should be noted that the delay functions used here are bounded and satisfy Assumption 1 with  $\Upsilon_{ij} = \Delta_{ij}^\tau = 0.15$ , for all  $i, j \in \mathcal{N}$ .

Also, the control gains are selected as:  $\alpha_0 = 1$ ,  $\alpha_1 = 2$ ,  $k_i^s = 15$ ,  $\Pi_i = 0.3\mathbf{I}_5$ ,  $\lambda_i = 10$ , and  $k_i^\eta = 2$ , for  $i \in \mathcal{N}$ . Note that this choice of the gains guarantees that the small-gain conditions (11) are satisfied for every system in the team. Fig. 1-2 show the obtained results, where it can be seen that all systems synchronize their positions despite the presence of irregular communication delays.

## VII. CONCLUSIONS

We addressed the synchronization problem of uncertain Euler-Lagrange systems in the presence of time-varying communication delays. Using the multi-dimensional IOS small gain approach, we proposed an adaptive control scheme that achieves synchronization under a directed communication

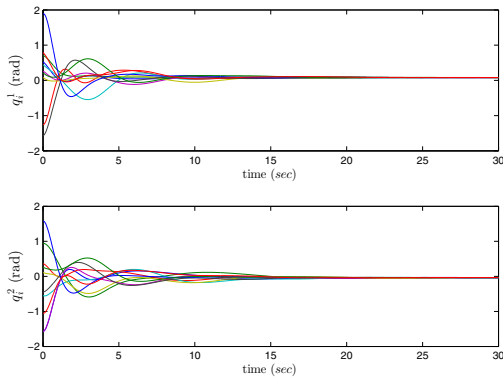


Fig. 1. Joint angles derivatives of the ten systems, with  $\mathbf{q}_i = (q_{i1}, q_{i2})^T$ .

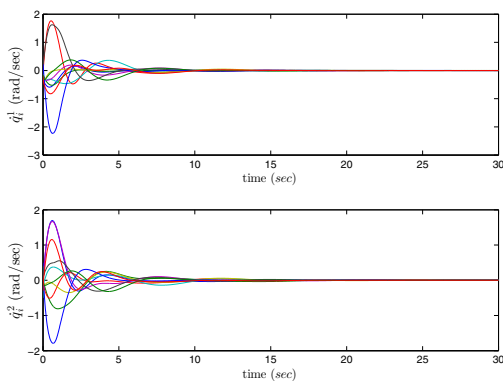


Fig. 2. Joint angles derivatives of the ten systems, with  $\dot{\mathbf{q}}_i = (\dot{q}_i^1, \dot{q}_i^2)^T$ .

graph containing a spanning tree. Essentially, we have extended the results of [16] to the adaptive case and the sufficient conditions for synchronization obtained in this paper can be easily satisfied by an appropriate choice of the control gains. To the best of our knowledge, the synchronization problem of Euler-Lagrange systems has been addressed in the literature only in the case of constant communication delays, which is a particular case of this work. In fact, our approach handles time-varying, possibly discontinuous, communication delays and does not impose constraints on the upper bound of the communication delays. Possible directions for future research include the extension of these results to the case of directed and switching communication topologies and significant information losses, as well as synchronization with prescribed nonzero velocities.

## APPENDIX

### A. Lemma

The following useful lemma can be found for example in [18] and [22]

*Lemma 1:* Consider an LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad (31)$$

where  $\mathbf{x} \in \mathbb{R}^n$  is the state vector,  $\mathbf{u} \in \mathbb{R}^m$  is the input vector, and  $\mathbf{A}$  is a Hurwitz matrix such that  $\mathbf{A} + \mathbf{A}^* + 2\mu\mathbf{I} \leq$

0 for some  $\mu > 0$ . Then for any initial condition  $\mathbf{x}(t_0)$ , the solution of (31) satisfies:  $|\mathbf{x}(t)| \leq e^{-\mu(t-t_0)}|\mathbf{x}(t_0)| + \frac{\|\mathbf{B}\|}{\mu} \sup_{\sigma \in [t_0, t]} |\mathbf{u}(\sigma)|$ .

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