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# Direct time injection in the loop: A new adaptive control point of view

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***Abstract*— In this paper, we show that adaptive control can be traded off against time-varying static state feedback (TVSSF) for a certain class of uncertain nonlinear systems. More precisely, we propose a systematic design procedure leading to TVSSF guaranteeing asymptotic (exponential) stability. Our approach, coined *Direct time injection in the loop* (DTIL), allows to remove the need for the integral action which is known to be instrumental in classical adaptive control design. The time-varying functions involved in the control law, allow an explicit selection of the convergence rates. Hence, exponential stability can be achieved without any requirement such as the persistency of excitation (PE). Interestingly, under certain conditions, the unknown parameters can be identified through a static function of the states and time.**

## I. Introduction

Adaptive control is one of the most studied techniques in control theory, with a long-lasting history, that has been and still is fascinating the control community (see, for instance, [3], [4], [5], [6], [8], [10], [13], [14], [15]). We believe that it is fair to say that, adaptive control sparked the development of important and powerful tools in control theory, that would probably not have seen the day without the theoretical and practical challenges that this technique brought to life.

Roughly speaking, among the rich literature related to adaptive control, one can distinguish the following two major classes. The first class is known as identifier-based adaptive control, relying on a parameterized control input and an identifier or adaptation law [1]. The second class is a non-identifier based high-gain universal adaptive control, which does not rely on any parametric identification, but rather uses the fundamental principle of the high gain, where the gain is obtained adaptively (see for instance, [3], [7], [9], [16]).

A common feature of the above mentioned techniques, which is the essence of adaptive control, is the use of a dynamic adaptation law or a *search function*, which is usually an integral action of the output and/or the states[2]. It is well known that, classical adaptive control laws lead generally to asymptotic stability. Exponential stability (which is the most sought-for property in practical control systems) is rarely achievable even under the persistency of excitation property. It is also well known that exponential stability is a desirable feature, since exponentially stable systems, beside their high convergence rates, exhibit a certain tolerance to modeling uncertainties, noise and slow parameters variations [1], ([13], Section 5.3).

In this paper, we present a new adaptive control point of view, where we remove the classical integral-based identifier through a direct time injection in the loop, leading to time-varying static state feedback control laws, with *controllable* convergence rates (*i.e.,* the desired convergence rate can be explicitly specified through the time-functions involved in the feedback). Of course, this approach comes with its theoretical and practical limitations that will be partially addressed in this paper, but at least it sets (in our opinion) the foundation for a new perspective in the adaptive control field.

It is worth mentioning that smooth time-varying controllers have been extensively used in the literature related to non-holonomic systems (see, for instance, [11], [12]) since the celebrated paper [2] on the non-existence of smooth time-invariant state feedback for the stabilization of driftless nonlinear systems. However, to the best of our knowledge, TVSSF has not been applied in our present paper's context of adaptive control design.

To motivate our contribution, let us take a look at the following scalar system:

$$\dot{x} = \theta x + u \tag{1}$$

where $\theta$ is constant and unknown. It is well known that the following *certainty equivalence* based adaptive controller

$$\begin{aligned} u(x,\hat{\theta}) &= -k_1 x - \hat{\theta} x \\ \dot{\hat{\theta}} &= k_2 x^2, \end{aligned} \tag{2}$$

with $k_1, k_2 > 0$, guarantees global asymptotic stability (GAS) of the equilibrium point $x = 0$.

Roughly speaking, the integral term $\hat{\theta} = \int_0^t k_2 x^2(\tau) d\tau$ is designed to *search* for the appropriate value (which is equal to $\theta$ if the PE condition is satisfied), that will generate a negative closed loop gain $\left(-k_1 + \theta - \int_0^t k_2 x^2(\tau) d\tau\right)$.

Now, a question that may arise is whether it is possible to find an explicit function of the state and time, $\eta(x,t)$, (without integral action) that does a similar job as the integral term $\hat{\theta}$ in terms of searching for the appropriate closed loop gain. In other words, can we find a TVSSF $u(x,t) = -k_1 x - \eta(x,t) x$ guaranteeing global asymptotic (exponential) stability of the equilibrium point $x = 0$. The answer to this question is yes (for this particular case and for a certain class of nonlinear systems). One solution to this problem (which is obtained through a constructive procedure that will be detailed later) is given by the following TVSSF guaranteeing

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[1]In this class there are several variants such as the classical *certainty equivalence* based approaches ( see, for instance, [4], [13], [8]), as well as *non-certainty equivalence* approaches such as [10].

[2]In some cases, such as in [10], the parametric adaption uses an additional state-dependant nonlinear term on top of the integral action

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global exponential stability

$$u = -k_1(t)x - \eta(x,t)x, \tag{3}$$

with $\eta(x,t) = \frac{1}{2}k_2(t)x^2$, $k_2(t) = \int_0^t \delta(\tau)d\tau$, $k_1(t) = \frac{\delta(t)}{2k_2(t)}$, where $\delta(t) > \frac{1}{2}$ for all $t \in \mathbb{R}_{\geq 0}$. One simple choice, for the function $\delta$ is $\delta(t) = 2\alpha\beta e^{2\alpha t}$, with $\alpha\beta > \frac{1}{4}$, leading to $k_1 = \alpha > 0$ and $k_2 = \beta e^{2\alpha t}$.

Interestingly, under the control law (3), we have $\dot{\tilde{\theta}} = -k_2 x^2 \tilde{\theta}$, with $\tilde{\theta} = \theta - \eta$. Therefore, the function $\eta(x,t)$ plays the role of an *identifier* and converges to the unknown parameter $\theta$ if $\theta > 0$. The restriction ($\theta > 0$) is due to the fact that $\eta(x,t) \geq 0$ in this particular case.

Another interesting feature of the TVSSF (3), that will become clear later, is that it achieves global exponential stability regardless of the richness of the signal $x$.

In this paper, we show the feasibility of our approach for a class of uncertain nonlinear systems, and through some simple academic examples we show that our controllers outperform (in some cases) the classical adaptive control laws. Unfortunately, it is not clear at this point in time whether our approach could be generalized to a broader class of multidimensional nonlinear systems with unmatched uncertainties.

## II. A CLASS OF UNCERTAIN NONLINEAR POLYNOMIAL SYSTEMS

First, to illustrate our approach, we consider the following polynomial system

$$\dot{x} = \theta x^n + u, \tag{4}$$

where $n$ is a non-negative integer, $x \in \mathbb{R}$ is the state, $u \in \mathbb{R}$ the control input. The parameter $\theta$ is constant and unknown.

### A. Control design

Consider the following Lyapunov function candidate

$$V(x,\tilde{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}\tilde{\theta}^2, \tag{5}$$

where $\tilde{\theta} = \theta - \eta(x,t)$. The time-derivative of (5), in view of (4), is given by

$$\dot{V} = x(\theta x^n + u) + \tilde{\theta}\left(-\frac{\partial \eta}{\partial x}(\theta x^n + u) - \frac{\partial \eta}{\partial t}\right) \tag{6}$$

At this point, we proceed as in the *certainty equivalence* based adaptive control, *i.e.,* set the control input to generate a negative quadratic term ($-k_1x^2$) and a residual term depending on $\tilde{\theta}$, namely ($\tilde{\theta}x^{n+1}$). To this end, let

$$u = -k_1(t)x - \eta(x,t)x^n, \tag{7}$$

which leads to

$$\dot{V} = -k_1x^2 + \tilde{\theta}x^{n+1} + \tilde{\theta}\left(-\frac{\partial \eta}{\partial x}(\tilde{\theta}x^n - k_1x) - \frac{\partial \eta}{\partial t}\right) \tag{8}$$

In classical certainty equivalence based adaptive control, the adaptive law is designed to cancel out the *unwanted* cross term $\tilde{\theta}x^{n+1}$. In our approach, we don't cancel it, we rather dominate it with $\frac{\partial \eta}{\partial x}$ to generate a positive quadratic polynomial in terms of $\tilde{\theta}$ and $x$, leading to $\dot{V} \leq 0$. The term $\frac{\partial \eta}{\partial t}$ will be designed to eliminate the *unwanted* term $k_1\frac{\partial \eta}{\partial x}x$. In fact, picking

$$\frac{\partial \eta}{\partial x} = k_2(t)x^n, \tag{9}$$

$$\frac{\partial \eta}{\partial t} = k_1(t)k_2(t)x^{n+1}, \tag{10}$$

the Lyapunov time-derivative (8) becomes

$$\dot{V} = -x^2(k_2z^2 - z + k_1) \tag{11}$$

with $z = \tilde{\theta}x^{n-1}$. Note that $k_2z^2 - z + k_1 > 0$ provided that $k_2 > 0$ and $k_1k_2 > \frac{1}{4}$.

Now, from (9), one can find a solution for $\eta$ as follows:

$$\eta(x,t) = \frac{1}{n+1}k_2(t)x^{n+1}. \tag{12}$$

To satisfy (10), the following condition must hold

$$\frac{dk_2(t)}{dt} = (n+1)k_1(t)k_2(t). \tag{13}$$

### B. Stability analysis

**Case $n \geq 1$:**

Let us assume that $k_1(t) > 0$ and $k_2(t) > \frac{1}{4k_1(t)}$ for all $t \in \mathbb{R}_{\geq 0}$. With this choice, we make sure that $k_2z^2 - z + k_1 > 0$. It is clear that (11) is negative semi-definite, and hence, one can conclude that $x$ and $\tilde{\theta}$ are bounded, from which the boundedness of $\eta(x,t)$ follows. Now, let us show that $\dot{V}$ is uniformly continuous as long as $k_1(t)$ and $\dot{k}_1(t)$ are bounded (regardless of $k_2(t)$ which might be unbounded). To this end, let us evaluate $\ddot{V}$.

$$\begin{aligned}\ddot{V} \;\; = \;\; & -\dot{k}_1x^2 + (\tilde{\theta}x^n - k_1x)(-2k_1x + (n+1)\tilde{\theta}x^n \\ & -2nk_2\tilde{\theta}^2x^{2n-1}) - k_2\tilde{\theta}x^{3n+1} + 2k_2^2\tilde{\theta}^2x^{4n}.\end{aligned} \tag{14}$$

Since $x$, $\tilde{\theta}$, and $k_2x^{n+1}$ are bounded, it is clear that $\ddot{V}$ is bounded as long as $k_1(t)$ and $\dot{k}_1(t)$ are bounded.

Finally, invoking Barbalat Lemma, one can conclude that $x(t)$ goes to zero when $t$ tends to infinity.

**Case $n = 0$:**

In the case where $n = 0$, *i.e.,* $\dot{x} = \theta + u$, the Lyapunov time-derivative (11) becomes

$$\dot{V} = -k_1x^2 - k_2\tilde{\theta}^2 + \tilde{\theta}x, \tag{15}$$

which is negative definite if $k_1(t) > 0$ and $k_2(t) > \frac{1}{4k_1(t)}$ for all $t \in \mathbb{R}_{\geq 0}$. Therefore, it is clear that $x$ and $\eta(x,t)$ are bounded. In this case, to prove the convergence of $x(t)$ to zero, it is not suitable to use the sufficient condition for uniform continuity in terms of the boundedness of $\ddot{V}$, since it requires to the boundedness of both $k_1(t)$ and $k_2(t)$. However, one can look at the closed loop system which is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} -k_1(t) & 1 \\ 0 & -k_2(t) \end{bmatrix} \begin{bmatrix} x \\ \tilde{\theta} \end{bmatrix} \tag{16}$$

from which, one can conclude that $\tilde{\theta}(t) = \tilde{\theta}(0)e^{-\int_0^t k_2(\tau)d\tau}$. Therefore, it is clear that $\tilde{\theta}$ and $x(t)$ converge exponentially to zero, if $k_1(t)$ and $k_2(t)$ satisfy the following condition

$$\int_t^{t+T} k_i(\tau)d\tau \geq \alpha T, \quad \forall t \geq 0 \tag{17}$$

for some $\alpha > 0$ and $T > 0$, $i = 1,2$.

### C. Convergence rates

Since $\eta(x,t)$ is bounded, it is clear that there exists a positive constant $c$ such that $|k_2(t)x^{n+1}(t)| \leq c$. Hence, $|x^{n+1}(t)| \leq \frac{c}{k_2(t)}$. Consequently, one can achieve any convergence rate through the choice of $k_2(t)$.

*Remark 1:* There are numerous possibilities with regards to the choice of $k_1$ and $k_2$. One of them consists of picking $k_2(t) = \frac{\alpha}{(n+1)k_1(t)} = \alpha t + \gamma$, with $\alpha > \frac{n+1}{4}$ and $\gamma > 0$. This leads to the following TVSSF

$$u = -\frac{\alpha}{(n+1)(\alpha t + \gamma)}x - \frac{1}{n+1}(\alpha t + \gamma)x^{2n+1}, \tag{18}$$

with $\alpha > \frac{n+1}{4}$ and $\gamma > 0$, making the equilibrium point $x = 0$ GAS. The convergence rates achieved with this choice are low, for $|x(t)| \leq c(\alpha t + \gamma)^{-1/(n+1)}$, $c > 0$. In the case $n = 0$, this choice is not guaranteed to work since (17) is violated. A second possibility consists of picking $k_1(t) = \alpha > 0$, $k_2 = \beta e^{(n+1)\alpha t}$, with $\beta > \frac{1}{4\alpha}$. This choice results in the following TVSSF

$$u = -\alpha x - \beta e^{(n+1)\alpha t}x^{2n+1}, \tag{19}$$

making the equilibrium point $x = 0$ globally exponential stable (GES). In this case we achieve exponential convergence rate as $|x(t)| \leq \frac{c}{\beta}e^{-\alpha t}$, $c > 0$. In this case, all the required conditions on $k_1$ and $k_2$ are satisfied including (17).

## III. A MORE GENERAL CASE

Now, let us consider the following system

$$\dot{x} = \theta\xi(x) + u, \tag{20}$$

where $\theta$ is an unknown constant parameter, and $\xi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, and well defined over the domain $D_\xi \subseteq \mathbb{R}$ containing the origin.

Our result is stated in the following theorem:

*Theorem 1:* Consider system (20), under the following TVSSF

$$u(x,t) = -k_1(t)\gamma(x) - \eta(\psi(x),t)\,\xi(x) \tag{21}$$

with

$$\eta(\psi(x),t) = \frac{k_2(t)}{p+1}\psi^{p+1}(x) \tag{22}$$

where $p$ is a non-negative integer. Let $k_1(t)$ be a positive bounded function of time with a bounded first time-derivative (in the case $p \geq 1$). The function $k_2(t)$ is solution of $\dot{k}_2(t) = (p+1)k_1(t)k_2(t)$, with $k_1(t)k_2(t) > \frac{1}{4}$. In the case $p = 0$, let $k_1(t)$ and $k_2(t)$ satisfy (17). The scalar function $\psi : D_\psi \to \mathbb{R}$ is continuously differentiable over the domain $D_\psi \subseteq D_\xi$ containing the origin, and satisfies

$$\frac{\partial\psi(x)}{\partial x}\xi(x) = \psi^p(x), \tag{23}$$

as well as the condition ($\psi(x) = 0 \;\Rightarrow\; x = 0$). The scalar function $\gamma : D_\gamma \to \mathbb{R}$ is given by

$$\gamma(x) = \frac{\xi(x)}{\psi^{p-1}(x)} \tag{24}$$

and is assumed to be well defined over the domain $D_\gamma \subseteq D_\xi$ containing the origin. Then,

*i)* $x = 0$ is asymptotically stable.

*ii)* There exists a positive constant $c$, such that $|\psi(x)| \leq c(k_2(t))^{-1/(p+1)}$.

*iii)* If $p$ is even (including $p = 0$), $\psi(-x) \neq \psi(x)$, and the following PE condition is satisfied

$$\int_t^{t+T} k_2(\tau)\psi^{2p}(x(\tau))d\tau \geq \alpha T, \quad \forall t \geq 0 \tag{25}$$

for some $\alpha > 0$ and $T > 0$, then $\lim_{t\to\infty}\eta(x,t) = \theta$.

*iv)* If $p$ is odd, and (25) is satisfied, then $\lim_{t\to\infty}\eta(x,t) = \theta$ if $\theta > 0$.

Note that in the case where $D_\psi \cap D_\gamma \cap D_\xi = \mathbb{R}$, and $\lim_{|x|\to\infty}\psi^2(x) = \infty$ (radially unbounded), the results are global.

*Proof:* Let us consider the following Lyapunov function candidate

$$V(x,\tilde{\theta}) = \frac{1}{2}\psi^2(x) + \frac{1}{2}\tilde{\theta}^2, \tag{26}$$

where $\tilde{\theta} = \theta - \eta(\psi,t)$. The time-derivative of (26), in view of (20), is given by

$$\dot{V} = \psi\frac{\partial\psi}{\partial x}(\xi\theta + u) + \tilde{\theta}\left(-\frac{\partial\eta}{\partial\psi}\frac{\partial\psi}{\partial x}(\xi\theta + u) - \frac{\partial\eta}{\partial t}\right), \tag{27}$$

which, under the control law (21), becomes

$$\begin{aligned}\dot{V} \;\;=\;\; & -k_1\psi\frac{\partial\psi}{\partial x}\gamma + \psi\frac{\partial\psi}{\partial x}\xi\tilde{\theta} \\ & +\tilde{\theta}\left(-\frac{\partial\eta}{\partial\psi}\frac{\partial\psi}{\partial x}(\xi\tilde{\theta} - k_1\gamma) - \frac{\partial\eta}{\partial t}\right).\end{aligned} \tag{28}$$

Taking

$$\frac{\partial\eta(\psi,t)}{\partial\psi} = k_2(t)\psi^p, \tag{29}$$

$$\frac{\partial\eta(\psi,t)}{\partial t} = k_1(t)k_2(t)\psi^{p+1}, \tag{30}$$

and using (23) and (24), and the fact that $\frac{\partial\psi}{\partial x}\gamma = \psi$, (28) becomes

$$\dot{V} = -\psi^2(x)(k_1(t) - z(x,t) + k_2(t)z^2(x,t)), \tag{31}$$

with $z(x,t) = \psi^{p-1}(x)\tilde{\theta}$.

Note that $k_1(t) - z(x,t) + k_2(t)z^2(x,t) > 0$ as long as $k_1(t) > 0$ and $4k_1(t)k_2(t) > 1$. The function $\eta(x,t)$ given in (22) is obtained from (29) and (30) under the constraint $\dot{k}_2(t) = (p+1)k_1(t)k_2(t)$.

Since $\dot{V}$ is negative semi-definite, one can conclude that $V(x,\tilde{\theta})$ is non-increasing and hence $\psi^2(x) \leq \psi^2(x) + \tilde{\theta}^2(x,t) \leq \psi^2(x(0)) + \tilde{\theta}^2(x(0),0)$. Therefore, there exists a compact domain $D_0 \subseteq D_\psi \cap D_\gamma$ containing the origin, such that all trajectories starting in $D_0$ remain in this domain for all subsequent times. To show that $\lim_{t\to\infty}\psi(x(t)) = 0$, we need to show that $\dot{V}$ is uniformly continuous.

The second time-derivative of (26) is given by

$$\begin{aligned}\ddot{V} \;\;=\;\; & (\tilde{\theta}\psi^p - k1\psi)(-2k_1\psi - 2pk_2\psi^{2p-1}\tilde{\theta}^2 \\ & +(p+1)\tilde{\theta}\psi^p) - \dot{k}_1\psi^2 \\ & -k_2\tilde{\theta}\psi^{3p+1} + 2k_2^2\tilde{\theta}^2\psi^{4p}.\end{aligned} \tag{32}$$

Since $\psi(x)$ and $k_2(t)\psi^{p+1}$ are bounded, from (32), one can conclude that $\ddot{V}$ is bounded as long as $k_1(t)$ and $\dot{k}_1(t)$ are bounded. This implies that $\dot{V}$ is uniformly continuous and hence $\dot{V}(x,t) \to 0$ as $t$ goes to infinity, which guarantees that $\lim_{t\to\infty} \psi(x(t)) = 0$ and consequently $\lim_{t\to\infty} x(t) = 0$.
In the case where $p = 0$, we take another route for the proof. In fact, the closed loop system is given by

$$\begin{bmatrix} \dot{\psi} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} -k_1 & \psi^p \\ 0 & -k_2\psi^{2p} \end{bmatrix} \begin{bmatrix} \psi \\ \tilde{\theta} \end{bmatrix} \tag{33}$$

which, in the case $p = 0$, leads to $\tilde{\theta}(t) = \tilde{\theta}(0)e^{-\int_0^t k_2(\tau)d\tau}$. Therefore, it is clear that $\tilde{\theta}$ (and consequently $\psi(x)$) converge exponentially to zero, if $k_1(t)$ and $k_2(t)$ satisfy condition (17). Since $k_2(t)\psi^{p+1}$ is bounded, it is clear that there exists a positive constant $c$ such that $|k_2(t)\psi^{p+1}(t)| \le c$, that is, $|\psi| \le ck_2^{\frac{-1}{p+1}}$.
Now, to prove the last two claims, we consider the closed loop system (33), from which one can clearly see that $\dot{\tilde{\theta}} = -k_2\psi^{2p}\tilde{\theta}$, hence the convergence of $\tilde{\theta}$ to zero is guaranteed under condition (25). However, there is an obstruction to the convergence of $\eta(\psi(x),t)$ to $\theta$ in the case where $p$ is odd since $\eta(x,t)$ cannot take negative values, an hence cannot converge to a negative parameter $\theta$. ■

*Remark 2:* It is clear that the control law proposed in Section II for the polynomial system (4), is a particular case of Theorem 1, with $\xi(x) = x^n$. In this case, setting $p = n$, we retrieve $\psi(x) = x$, $\gamma(x) = x$ and $\eta(x,t) = \frac{1}{n+1}k_2(t)x^{n+1}$. Since $\psi(x)^2 = x^2$ is radially unbounded, in this case $x = 0$ is GAS.

*Remark 3:* One simple solution that satisfies all the requirements on $k_1$ and $k_2$ in Theorem 1, is given by $k_1(t) = \alpha > 0$, $k_2 = \beta e^{(p+1)\alpha t}$, with $\beta > \frac{1}{4\alpha}$. This choice leads to exponential stability as $|x(t)| \le \frac{c}{\beta}e^{-\alpha t}$, $c > 0$.

*Remark 4:* In practice, the system model might be inaccurate due to the presence of measurement noise, external disturbances and unmodeled dynamics, that might cause $\eta(x,t)$ to drift since the gain $k_2(t)$ is an increasing function of time. Therefore, in practical applications, one can use some of the robustification techniques commonly used in classical adaptive control such as the dead-zone, by *freezing* the function $\eta(x,t)$ when an acceptable level of performance is reached, and resetting the *time t* each time $\eta(x,t)$ is turned back on.

## IV. Examples

In this section, we present some simple academic examples to show the feasibility of our approach. In some cases, the results obtained with the TVSSF, clearly outperform the results obtained with the classical adaptive control approach. However, in some situation, where the classical adaptive control provides *global asymptotic stability*, our approach provides only *local exponential stability*.

### A. Example 1

Consider the following system

$$\dot{x} = \theta + u, \tag{34}$$

A straightforward application of Theorem 1, with $p = 0$, leads to the following TVSSF scheme

$$u = -k_1(t)x - \eta(x,t), \tag{35}$$

$$\eta(x,t) = k_2(t)x, \tag{36}$$

with $k_1(t) = \alpha > 0$, $k_2 = \beta e^{\alpha t}$ and $\beta > \frac{1}{4\alpha}$. Note that $k_1$ and $k_2$ satisfy (17). This choice results in the following control law

$$u = -\alpha x - \beta e^{\alpha t}x, \tag{37}$$

making the equilibrium point $(x = 0, \tilde{\theta} = 0)$ GES. In this case, $\lim_{t\to\infty} \eta(x,t) = \theta$.

### B. Example 2

Consider system (1), given in the Introduction. Applying Theorem 1, with $p = 1$, we obtain the following TVSSF scheme

$$u = -k_1(t)x - \eta(x,t)x, \tag{38}$$

$$\eta(x,t) = \frac{1}{2}k_2(t)x^2, \tag{39}$$

with $k_1(t) = \alpha > 0$, $k_2 = \beta e^{2\alpha t}$ and $\beta > \frac{1}{4\alpha}$. This choice results in the following control law

$$u = -\alpha x - \frac{\beta}{2}e^{2\alpha t}x^3, \tag{40}$$

which makes $x = 0$ GES, since $|x| \le \frac{c}{\beta}e^{-\alpha t}$, $c > 0$. Furthermore, if (25) is satisfied with $\psi(x) = x$, and $\theta > 0$ then $\lim_{t\to\infty} \frac{1}{2}k_2(t)x(t)^2 = \theta$.

### C. Example 3

Consider the following system

$$\dot{x} = \theta x^2 + u. \tag{41}$$

Again, a straightforward application of Theorem 1, with $p = 2$, leads to the following TVSSF scheme

$$u = -k_1(t)x - \eta(x,t)x^2, \tag{42}$$

$$\eta(x,t) = \frac{1}{3}k_2(t)x^3, \tag{43}$$

with $k_1(t) = \alpha > 0$, $k_2 = \beta e^{3\alpha t}$ and $\beta > \frac{1}{4\alpha}$. This choice results in the following control law

$$u = -\alpha x - \frac{\beta}{3}e^{3\alpha t}x^5, \tag{44}$$

which makes $x = 0$ GES, since $|x| \le \frac{c}{\beta}e^{-\alpha t}$, $c > 0$. Furthermore, if (25) is satisfied with $\psi(x) = x$, then $\lim_{t\to\infty} \frac{1}{3}k_2(t)x(t)^3 = \theta$. To illustrate the performance of our approach, we simulated (41) under the control law (44), with $x(0) = 1$, $\theta = 5$, $k_1 = 1$ and $\beta = 1$. We also simulated (41) under the following classical adaptive control law

$$\begin{aligned} u(x,\hat{\theta}) &= -k_1x - \hat{\theta}x^2 \\ \dot{\hat{\theta}} &= k_2x^3, \end{aligned} \tag{45}$$

with $k_1 = k_2 = 1$ and $\hat{\theta}(0) = 0$. The results are shown in Fig. 1, Fig. 2 and Fig. 3, where we can clearly see that the parameter estimate $\hat{\theta}$ of the classical adaptive control does not converge to the real parameter $\theta = 5$, whereas the parameter estimate $\eta$ of the TVSSF does.

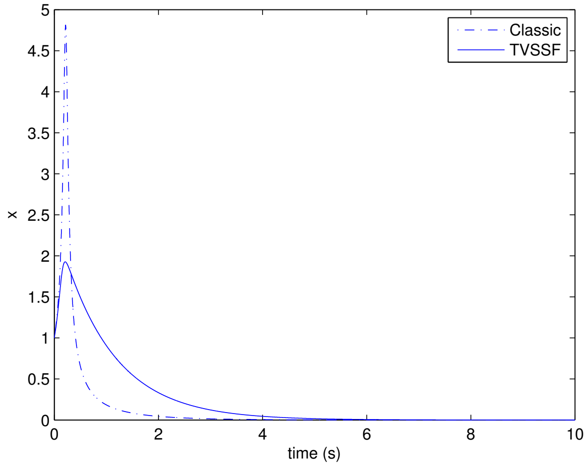


Fig. 1. Example 3: State variable vs. time for the classical adaptive control and the TVSSF

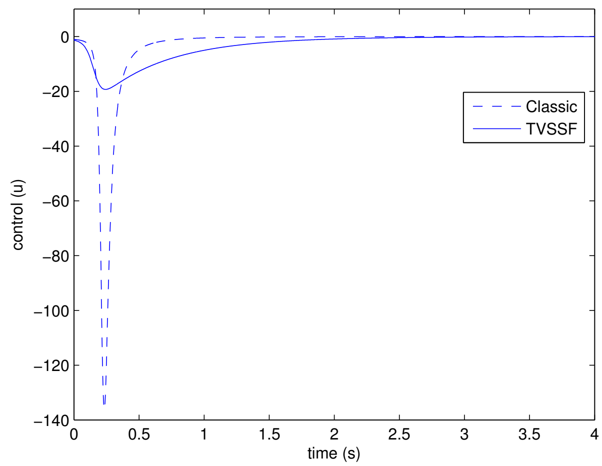


Fig. 3. Example 3: Control input vs. time for the classical adaptive control and the TVSSF

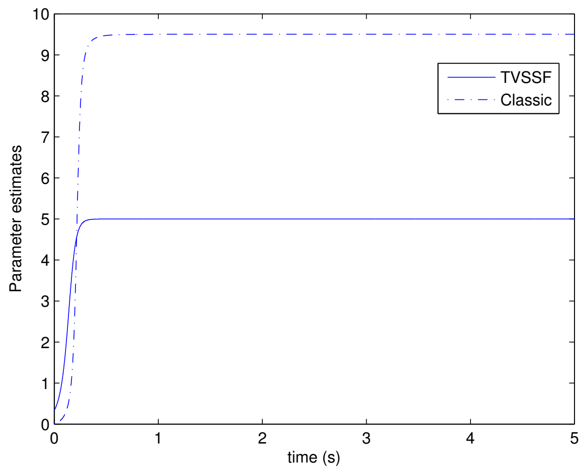


Fig. 2. Example 3: Parameter estimate vs. time for the classical adaptive control and the TVSSF

with $k_1(t) = \alpha > 0$, $k_2 = \beta e^{3\alpha t}$ and $\beta > \frac{1}{4\alpha}$. The equilibrium point $x = 0$ is locally exponentially stable and an estimate of the domain of attraction is given by $D_0 = \{x \in \mathbb{R} \mid \psi(x)^2 \leq \sigma\}$ for some finite $\sigma > 0$. We can show that $D_0$ is actually $(-\frac{\pi}{2}, \frac{\pi}{2})$.

We simulated (46) under the control law (49), with $x(0) = 1$, $\theta = 10$, $k_1 = 1$ and $\beta = 1$. We also simulated (46) under the following classical adaptive control law

$$\begin{aligned} u(x,\hat{\theta}) &= -k_1 x - \hat{\theta}\sin^2 x \\ \dot{\hat{\theta}} &= k_2 x \sin^2 x, \end{aligned} \tag{50}$$

with $k_1 = k_2 = 1$ and $\hat{\theta}(0) = 0$. The results are shown in Fig. 4, Fig. 5 and Fig. 6, where we can clearly see that the parameter estimate $\hat{\theta}$ of the classical adaptive control does not converge to the real parameter $\theta = 10$, whereas the parameter estimate $\eta$ of the TVSSF does.

### *D. Example 4*

Consider the system

$$\dot{x} = \theta \sin^2 x + u. \tag{46}$$

In this example $\xi(x) = \sin^2 x$. To find $\psi(x)$, we set $p = 2$, and solve

$$\frac{\partial \psi}{\partial x}\sin^2 x = \psi^2, \tag{47}$$

that is

$$\int \frac{d\psi}{\psi^2} = \int \frac{dx}{\sin^2 x}, \tag{48}$$

which leads to $\psi(x) = \tan x$. The function $\gamma(x)$ is given by $\gamma(x) = \frac{\xi(x)}{\psi(x)} = \sin x \cos x$. Note that $\psi$ is well defined over the domain $D_\psi = (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\gamma$ is well defined over $\mathbb{R}$. The TVSSF is given by

$$u(x,t) = -k_1(t)\sin x \cos x - \frac{k_2(t)}{3}\tan^3 x \sin^2 x, \tag{49}$$

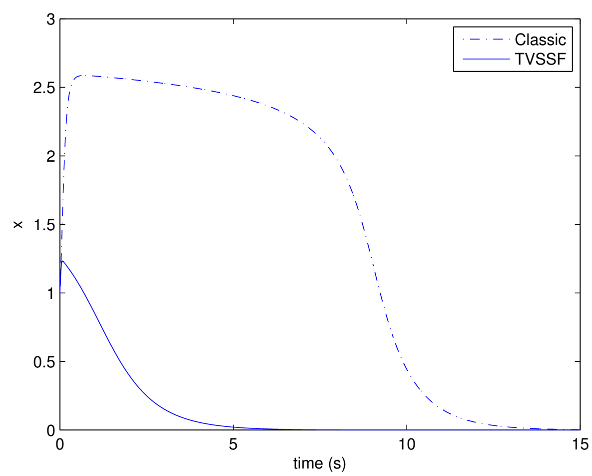


Fig. 4. Example 4: State variable vs. time for the classical adaptive control and the TVSSF

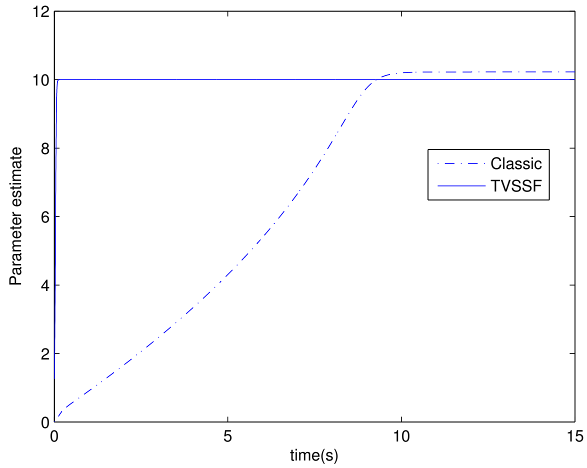


Fig. 5. Example 4: Parameter estimate vs. time for the classical adaptive control and the TVSSF

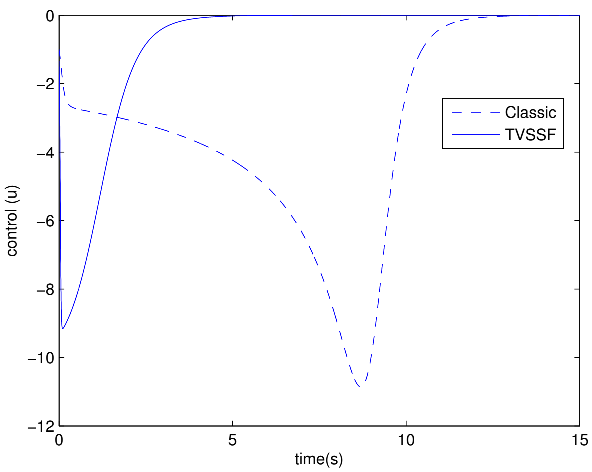


Fig. 6. Example 4: Control input vs. time for the classical adaptive control and the TVSSF

## V. Conclusion

In this work it is shown that time-varying static state feedback could be a possible alternative to adaptive control for a certain class of uncertain nonlinear systems. In fact, aside from its practical implication, being able to trade off the classical integral action which is the essence of adaptive control, against the DTIL is an interesting conceptual fact, which leads to a new point of view of adaptive control. Interestingly, the TVSSF resulting from the proposed approach allows the designer to specify the desirable convergence rates regardless of the frequency content (richness) of the signals. Moreover, in certain situations where classical adaptive control fails to identify the unknown parameters, the DTIL approach succeeds. Finally, on the negative side, unfortunately the class of systems for which the DTIL approach is applicable, seems to be limited at this point in time. In some cases, where classical adaptive control leads to global asymptotic stability, the DTIL provides only local exponential stability, and sometimes even fails to provide local results. Finally, since system robustness has not been addressed, the results presented here should be regarded as an initial conceptual contribution to the design of static adaptive controllers. Since the present control strategy involves an explicit high gain in the loop ($k_2(t)$ is an increasing function of time), robustness is likely to be a particularly interesting issue. On the ambitious side, one might think of the extension of this work to a wider class of nonlinear uncertain systems with non-matched uncertainties, as well as the design of time-varying static state observers.

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