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Adaptive position tracking of VTOL UAVs

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Abstract— **An adaptive position tracking control scheme is proposed for vertical thrust propelled unmanned airborne vehicles (UAVs) in the presence of external disturbances. As an intermediary step, the system attitude is used to direct the thrust towards the position target. Instrumental in our control design, an extraction method allowing to obtain the desired attitude and thrust from the required force driving the system towards the desired position, is proposed. Finally, the control torque is designed for the overall system to achieve the tracking objective. The proposed controller ensures global asymptotic stability of the overall closed loop system.**

I. Introduction

This paper deals with the position tracking problem of a vertical take-off and landing (VTOL) unmanned airborne vehicle in the presence of external disturbances. For this class of underactuated mechanical systems, a common practice is to use the system attitude as a means to direct the thrust in order to control the system position and velocity. This is an intuitive choice, which offers promising results when used with the backstepping approach. However, despite the tremendous efforts of the research community, there are still some open-problems in terms of handling the external disturbances, coupling between the translational and rotational dynamics, singularities as well as achieving global stability results. Examples of models including disturbance forces include [1], [2] and [3], where it is normally assumed that the disturbance force is constant, as is the case in this paper. However, future work may consider the fact that the disturbance is time-varying, especially if it is due to aerodynamic drag. A second common problem is related to which system inputs are used to define the control. In most cases, it is desired to obtain the control in terms of the torque applied to the rotational dynamics of the system. However, this becomes difficult, especially when the effect of external disturbances is included. The only case that we know of that expresses the control in terms of the torque, and deals with external disturbances, is in [3]. However, this result is local and deals with the regulation problem rather than tracking. Thirdly, a well known problem is due to the coupling between the rotational and translational dynamics. This coupling is usually in the form of the control torque or angular velocity acting on the rotational dynamics. This problem is well known and explored in more detail in [4]. Note that this coupling is system dependant, and is not always present in certain systems, for example the quad-rotor aircraft. A linear position tracking controller for such an aircraft is discussed in [5]. As is the case in our paper, it is often assumed that the coupling is negligible and is thus omitted in the control design. However, as discussed in [3] and [4], depending on the strength of the coupling, this can lead to unexpected oscillations in the system states. There are some examples of control which do address the coupling problem. For instance, in [6], a nice change of coordinates is presented, however, only for a planar system. In [3], a change of coordinates is also presented that removes the coupling due to the control torque. A consequence of this change of coordinates is that a new coupling is introduced in terms of the system angular velocity, which can only be neglected if the system yaw rate is assumed to be zero, which is the case in [3]. However, in practice this would likely not be the case. Therefore, there still seems to be some potential room for improvement regarding this coupling term in future work. Lastly, to the best of our knowledge there are no results in the available literature achieving global asymptotic stability for the position tracking problem of UAVs in the presence of disturbance forces. Rather we have controllers achieving practical stability, such as in [7] and [8], or local stability, such as in [1] and [2].

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In this paper, a new method is presented for extracting the magnitude and direction of the thrust, in terms of unit-quaternion, from a given desired translational force. Usually, attitude extraction is a form of Wahba's problem, which is discussed in [9], where an expression for the system attitude is sought based on a set of vector measurements. In [10] an attitude extraction method is used that uses two pairs of vector observations from an accelerometer and magnetometer, that is used to recover a measurement of the actual system attitude. For the case where only one set of vector measurements is given, as is the case with our method, there are an infinite number of solutions for the attitude extraction. However, we present a method to solve this problem with *almost* no restrictions on the two vector measurements, except for a mild singularity that can be avoided. This method is beneficial since generally, solutions of Wahba's problem require singular value decomposition, least squares or some numerical method.

Relying on the new quaternion extraction method, we present an adaptive tracking controller using the torque as a control input. The proposed controller, which uses an adaptive estimation method using a projection mechanism [11], [12], achieves global asymptotic stability in the presence of

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a bounded constant disturbance force.

II. Dynamic Model and Problem Formulation

In this paper we use the well known 6-DOF equations for a rigid body to develop the control laws. To represent the system attitude, we make use of two forms of attitude representation, namely the rotation matrix and unit-quaternion. The rotation matrix is the map from the inertial frame to the body frame where $R \in SO(3) := \{R \in \mathbb{R}^{3\times 3};\ det(R) = 1;\ RR^T = R^T R = I_{3\times 3}\}$. The dynamics of the rotation matrix are $\dot{R} = -S(\Omega)R$, where $S(\cdot)$ is the skew-symmetric matrix such that $S(u)v = u \times v$, and $\Omega \in \mathbb{R}^3$ is the body-referenced angular velocity of the rigid-body. The system attitude can also be represented by the unit quaternion $Q = (q_0, q) \in \mathbb{Q} = \{\mathbb{R} \times \mathbb{R}^3 |\ \ \|\mathbb{Q}\| = 1\}$ where $R = I_{3\times 3} + 2S(q)^2 - 2q_0 S(q)$, which has the dynamics

$$\dot{Q} = \frac{1}{2} Q \odot \begin{pmatrix} 0 \\ \Omega \end{pmatrix} \tag{1}$$

where the operator $\odot$ represents quaternion multiplication, or equivalently given $Q, P \in \mathbb{Q}$ where $P = (p_0, p)$ we have

$$Q \odot P = \begin{pmatrix} p_0 q_0 - q^T p \\ q_0 p + p_0 q + S(q)p \end{pmatrix} \tag{2}$$

Using this representation, the well known rigid body model can be obtained from the principle of conservation of linear and angular momentum, which is given by

$$\dot{p} = v \tag{3}$$

$$\dot{v} = g\hat{z} - \frac{T}{m} R^T \hat{z} - \frac{1}{m\ell} R^T S(\hat{z}) u + \frac{1}{m} F_d \tag{4}$$

$$\dot{Q} = \frac{1}{2} \begin{pmatrix} -q^T \\ q_0 I_{3\times 3} + S(q) \end{pmatrix} \Omega \tag{5}$$

$$I_b \dot{\Omega} = -S(\Omega) I_b \Omega + \epsilon_M S(\hat{z}) R F_d + u \tag{6}$$

where $p, v \in \mathbb{R}^3$ denote the inertial referenced system position and velocity, respectively, m is the system mass, g is the gravitational acceleration, F_d is the inertial referenced constant disturbance force, I_b is the constant body-referenced inertia tensor, u is the control torque input, ℓ is the torque lever arm, T is the system thrust, $\hat{z} = (0, 0, 1)$ and ϵ_M is the lever arm that creates a disturbance torque due to F_d. The model for the disturbance force and torque is similar to [3], since we assume that the disturbance force is applied to a point on the body-referenced z axis at a distance ϵ_M away from the system center of gravity. Note that the coupling between the translational and rotational dynamics appears in the equation of $\dot{v}$ in the form of the control torque u. For the purposes of the control design, this term is neglected where we assume that $m\ell >> 1$. For convenience we define the unknown parameters $\theta_a = m^{-1} F_d$, $\theta_b = \epsilon_M F_d$ and thrust control input $u_t = m^{-1} T$. Using the model given by (3) through (5) our objective is to force the position p to track some time-varying reference $r(t)$, given it meets the following requirements:

Assumption 1: The second, third and fourth derivatives (wrt t) of the reference trajectory $r(t)$ are uniformly continuous. Furthermore, the second derivative of the reference trajectory is bounded such that $\|\ddot{r}\| \le \delta_r$ and $\hat{z}^T \ddot{r} < \delta_{rz} < g$.

Having defined the reference trajectory $r(t)$ this motivates the choice of the error signals

$$\widetilde{p} = p - r(t) \tag{7}$$

$$\widetilde{v} = v - \dot{r}(t) \tag{8}$$

To use the system velocity v as a virtual control we must use the system attitude R and thrust u_t as virtual inputs to the translational dynamics (3)-(4). Therefore, we define

$$\mu = g\hat{z} - u_t R^T \hat{z} \tag{9}$$

$$\widetilde{\mu} = \mu - \mu_d \tag{10}$$

where μ_d will be defined later in the control design. The use of μ as a virtual control requires the extraction of the thrust T and the attitude (R or Q) from μ_d. An attitude and thrust extraction method to this effect is presented in the next section. In order to achieve asymptotic stability, we must ensure that the control law we specify for μ_d is bounded. As a consequence, we make the following assumption on the disturbances θ_a and θ_b:

Assumption 2: The disturbances θ_a and θ_b are constant and bounded such that $\theta_a \in B_a := \{\theta_a \in \mathbb{R}^3;\ \|\theta_a\| < \delta_a < g\}$, $\theta_b \in B_b := \{\theta_b \in \mathbb{R}^3;\ \|\theta_b\| < \delta_b\}$.

To control the system rotational dynamics, in addition to the system thrust u we also specify a desired angular velocity Ω_d and the error signal

$$\widetilde{\Omega} = \Omega - \Omega_d \tag{11}$$

where the expression for Ω_d is defined later in the control design.

III. Attitude extraction

Let μ_d be the virtual control input that achieves the position tracking objective for the translational dynamics. In order to realize μ_d, the control input τ for the rotational dynamics should be designed to take care of driving the system attitude to the desired attitude. Therefore, we need to extract the thrust u_t and the desired attitude R_d or Q_d for a given μ_d satisfying

$$\mu_d = \text{col}\left(\mu_{d1},\ \mu_{d2},\ \mu_{d3}\right) = g\hat{z} - u_t R_d^T \hat{z} \tag{12}$$

Although, in general, there are an infinite number of solutions to (12), we propose one solution for this problem using the unit quaternion:

Lemma 1: Given μ_d and assuming that $\mu_d \notin L$,

$$L = \{\mu_d \in \mathbb{R}^3;\ \mu_d = (0, 0, \mu_{d3});\ \mu_{d3} \in [g, \infty)\} \tag{13}$$

then a solution for the system thrust and attitude, in terms of the unit quaternion $Q_d = (q_{d0}, q_d)$, that satisfies (12) is given by

$$u_t = \|g\hat{z} - \mu_d\| \tag{14}$$

$$q_{d0} = \left(\frac{1}{2}\left(1 + \frac{g - \mu_{d3}}{\|g\hat{z} - \mu_d\|}\right)\right)^{1/2} \tag{15}$$

$$q_d = \frac{1}{2\|g\hat{z} - \mu_d\| q_{d0}} \begin{pmatrix} \mu_{d2} & -\mu_{d1} & 0 \end{pmatrix}^T \tag{16}$$

Proof: The proof is omitted.

We now desire to find the dynamics of the solutions (15) and (16) with respect to the change in μ_d as stated in the following lemma:

Lemma 2: Given that μ_d is differentiable, and using the well known expression for quaternion dynamics given by

$$\dot{Q}_d = \frac{1}{2} Q_d \odot \begin{pmatrix} 0 \\ \beta \end{pmatrix} \quad (17)$$

a solution for the angular velocity β of the desired quaternion Q_d is given by

$$\beta = \Phi(\mu_d)\dot{\mu}_d \quad (18)$$

$$\Phi(\mu_d) = \frac{1}{u_t^2 c_1} \begin{pmatrix} -\mu_{d1}\mu_{d2} & -\mu_{d2}^2 + u_t c_1 & \mu_{d2} c_1 \\ \mu_{d1}^2 - u_t c_1 & \mu_{d1}\mu_{d2} & -\mu_{d1} c_1 \\ \mu_{d2} u_t & -\mu_{d1} u_t & 0 \end{pmatrix} \quad (19)$$

where $c_1 = u_t + g - \mu_{d3}$.

Proof: The proof is omitted, yet this result is found from direct albeit tedius differentiation of (14)-(16).

Having obtained the desired quaternion Q_d, we now define the attitude error $\widetilde{Q} = (\widetilde{q}_0, \widetilde{q})$, where

$$\widetilde{Q} = Q_d^{-1} \odot Q \quad (20)$$

where we recall that Q represents the actual system attitude. The dynamics of the error quaternion are obtained in light of (1) and (17)

$$\dot{\widetilde{Q}} = \frac{1}{2}\left(\widetilde{Q} \odot \begin{pmatrix} 0 \\ \Omega \end{pmatrix} - \begin{pmatrix} 0 \\ \beta \end{pmatrix} \odot \widetilde{Q} \right) \quad (21)$$

IV. Adaptive Estimation Using Projection

To avoid the singularity (13) the virtual control μ_d that will be defined must be bounded such that $\hat{z}^T \mu_d = \mu_{d3} < g$. To achieve asymptotic stability, the control μ_d must include some estimate of the disturbance force. Therefore, to ensure that μ_d remains bounded, we utilize projection based adaptive estimation based on the second assumption. Given the error $\widetilde{\theta}_p = \theta_p - \hat{\theta}_p$ and adaptive estimation law $\dot{\hat{\theta}}_p = \tau$, one differentiable projection algorithm defined in [12] is given by

$$\dot{\hat{\theta}}_p = \tau + \alpha(\hat{\theta}_p, \delta_p, \tau) \quad (22)$$

$$\alpha(\hat{\theta}_p, \delta_p, \tau) = -k_\alpha \eta_1 \eta_2 \hat{\theta}_p \quad (23)$$

$$k_\alpha = \left(2\left(\epsilon_\alpha^2 + 2\epsilon_\alpha \delta_p\right)^{n+1} \delta_p^2 \right)^{-1} \quad (24)$$

$$\eta_1 = \begin{Bmatrix} \left(\hat{\theta}_p^T \hat{\theta}_p - \delta_p^2\right)^{n+1} & \text{if } \hat{\theta}_p^T \hat{\theta}_p > \delta_p^2 \\ 0 & \text{otherwise} \end{Bmatrix} \quad (25)$$

$$\eta_2 = \hat{\theta}_p^T \tau + \left(\left(\hat{\theta}_p^T \tau\right)^2 + \delta_\alpha^2 \right)^{1/2} \quad (26)$$

where $\epsilon_\alpha > 0$, $\delta_\alpha > 0$, which has the properties $\|\hat{\theta}_p\| < \delta_p + \epsilon_\alpha$, $\widetilde{\theta}_p^T \alpha \geq 0$ and $\dot{\hat{\theta}}_p \in C^n$

V. Global Position tracking control

Focusing on the translational dynamics first, we recall the position error $\widetilde{p}$ and velocity error $\widetilde{v}$ from (7) and (8). Recall $\theta_a = m^{-1} F_d$, then from (4), (9) and (10) where we neglect the coupling term, the velocity dynamics become

$$\dot{v} = \widetilde{\mu} + \mu_d + \theta_a \quad (27)$$

where we see that for equilibrium solution $(\dot{v}, \widetilde{\mu}) = 0$, the control must have some non-zero steady state value of $\mu_d = -\theta_a$. This motivates the use of a adaptive estimate of θ_a which will be used in the control law μ_d. We denote this adaptive estimate as $\hat{\theta}_1$ and define the error

$$\widetilde{\theta}_1 = \theta_a - \hat{\theta}_1 - k_\theta z(\widetilde{v}) \quad (28)$$

where $k_\theta > 0$ and

$$z(u) := \left(1 + u^T u\right)^{-1/2} u \quad (29)$$

$$\phi(u) := \frac{\partial}{\partial u} z(u) = \left(1 + u^T u\right)^{-3/2} \left(I_{3\times 3} - S(u)^2\right) \quad (30)$$

where we note that $0 < \underline{\lambda}(\phi(u)) \leq \bar{\lambda}(\phi(u)) \leq 1$, $\forall u$, where $\underline{\lambda}(\cdot)$ and $\bar{\lambda}(\cdot)$ denote the smallest and largest eigenvalue of $(\cdot)$, respectively. The term $k_\theta z(\widetilde{v})$ in (28) is used to force the error $\widetilde{\theta}_1$ to zero when $t \to \infty$ in order to achieve asymptotic stability for the position error $\widetilde{p}$.

Differentiating (7), (8) and (28) along (3)-(4) we obtain the translational error dynamics

$$\dot{\widetilde{p}} = \widetilde{v} \quad (31)$$

$$\dot{\widetilde{v}} = \widetilde{\mu} + \mu_d + \theta_a - \ddot{r} = \widetilde{\mu} + \mu_d + \widetilde{\theta}_1 + \hat{\theta}_1 + k_\theta z(\widetilde{v}) - \ddot{r} \quad (32)$$

$$\dot{\widetilde{\theta}}_1 = -k_\theta \phi(\widetilde{v}) \widetilde{\theta}_1 + \tau_1 - \dot{\hat{\theta}}_1 \quad (33)$$

$$\tau_1 = -k_\theta \phi(\widetilde{v}) \left(\mu + \hat{\theta}_1 + k_\theta z(\widetilde{v}) - \ddot{r} \right) \quad (34)$$

From (32) we define the virtual control

$$\mu_d = \ddot{r} - \hat{\theta}_1 - K_p z(\widetilde{p}) - K_v z(\widetilde{v}) \quad (35)$$

where $K_p = k_1 \Gamma_v^{-1}$, $K_v = (k_v + k_\theta) I$, $k_1 > 0$, $k_v > 0$, $\Gamma_v = \Gamma_v^T > 0$, from which we obtain

$$\dot{\widetilde{v}} = f_{\widetilde{v}_1} + \widetilde{\theta}_1 = f_{\widetilde{v}_2} + \theta_a \quad (36)$$

$$f_{\widetilde{v}_1} = -K_p z(\widetilde{p}) - k_v z(\widetilde{v}) + \widetilde{\mu} \quad (37)$$

$$f_{\widetilde{v}_2} = -K_p z(\widetilde{p}) - K_v z(\widetilde{v}) + \widetilde{\mu} - \hat{\theta}_1 \quad (38)$$

where the two separate functions $f_{\widetilde{v}_1}$ and $f_{\widetilde{v}_2}$ are defined for convenience. It is useful to note that the error signal $\widetilde{\mu} = \mu - \mu_d$ can be expressed in terms of the vector part of the error quaternion $\widetilde{q}$ by

$$\widetilde{\mu} = 2 u_t R^T S(\bar{q}) \widetilde{q} \quad (39)$$

$$\bar{q} = S(\hat{z})\widetilde{q} + \widetilde{q}_0 \hat{z} \quad (40)$$

Using the expression for μ_d given by (35) we find the system thrust input u_t and the desired system attitude in terms of the unit quaternion Q_d using (14), (15) and (16), respectfully.

If we recall (17) and (18) it is now possible to define the desired attitude dynamics $\dot{Q}_d$ in terms of the angular velocity β by obtaining $\dot{\mu}_d$. Differentiating (35) in light of (31) and (32) we find

$$\begin{aligned} \dot{\mu}_d &= f_{\mu_d} + g_{\mu_d}\theta_a & (41)\\ f_{\mu_d} &= r^{(3)} - \dot{\hat{\theta}}_1 - K_p\phi(\widetilde{p})\widetilde{v} - K_v\phi(\widetilde{v})f_{\widetilde{v}_2} & (42)\\ g_{\mu_d} &= -K_v\phi(\widetilde{v}) & (43) \end{aligned}$$

which is used with (18) and (41) to obtain

$$\begin{aligned} \beta &= \Phi(\mu_d)h_\beta + \Phi(\mu_d)g_{\mu_d}\widetilde{\theta}_1 & (44)\\ h_\beta &= f_{\mu_d} + g_{\mu_d}\left(\hat{\theta}_1 + k_\theta z(\widetilde{v})\right) & (45) \end{aligned}$$

Applying the quaternion multiplication (2) to (21) we obtain the attitude error dynamics $\dot{\widetilde{Q}} = (\dot{\widetilde{q}}_0, \dot{\widetilde{q}})$

$$\begin{aligned} \dot{\widetilde{q}}_0 &= \frac{1}{2}\widetilde{q}^T(\beta - \Omega) & (46)\\ \dot{\widetilde{q}} &= \frac{1}{2}\left(\widetilde{q}_0(\Omega-\beta) + S(\widetilde{q})(\beta+\Omega)\right) & (47) \end{aligned}$$

Using the above results, we now consider the first Lyapunov function

$$\begin{aligned} V_1 &= k_1\left(1+\widetilde{p}^T\widetilde{p}\right)^{1/2} + \frac{1}{2}\widetilde{v}^T\Gamma_v\widetilde{v}\\ &\quad +2\gamma_q(1-\widetilde{q}_0) + \frac{1}{2\gamma_{\theta_1}}\widetilde{\theta}_1^T\widetilde{\theta}_1 & (48) \end{aligned}$$

where $\gamma_q > 0$, $\gamma_{\theta_1} > 0$, which we differentiate along (31)-(33) and (46) to find

$$\begin{aligned} \dot{V}_1 &= -\frac{k_\theta}{\gamma_{\theta_1}}\widetilde{\theta}_1^T\phi(\widetilde{v})\widetilde{\theta}_1 - k_v\widetilde{v}^T\Gamma_v z(\widetilde{v})\\ &\quad +\widetilde{q}^T\left(-2u_tS(\bar{q})R\Gamma_v\widetilde{v} + \gamma_q\left(\Omega - \Phi(\mu_d)h_\beta\right)\right)\\ &\quad +\widetilde{\theta}_1^T\left(\tau_2 - \gamma_{\theta_1}^{-1}\dot{\hat{\theta}}_1\right) & (49)\\ \tau_2 &= \gamma_{\theta_1}^{-1}\tau_1 + \Gamma_v\widetilde{v} - \gamma_q g_{\mu_d}^T\Phi(\mu_d)^T\widetilde{q}\\ &= -\frac{k_\theta}{\gamma_{\theta_1}}\phi(\widetilde{v})f_{\widetilde{v}_1} + \Gamma_v\widetilde{v} + \gamma_q\phi(\widetilde{v})K_v\Phi(\mu_d)^T\widetilde{q} & (50) \end{aligned}$$

If we recall from (11), $\widetilde{\Omega} = \Omega - \Omega_d$, the next step in the design procedure requires defining the virtual control law Ω_d from (49). From (42) we see that f_{μ_d} depends on the value of $\dot{\hat{\theta}}_1$. Since we intend to use projection for the adaptive estimate, we must define the adaptation law $\dot{\hat{\theta}}_1$ before proceeding further. Therefore, we define the first adaptation law as

$$\dot{\hat{\theta}}_1 = \gamma_{\theta_1}\left(\tau_2 + \alpha\left(\hat{\theta}_1, \delta_a + k_\theta, \tau_2\right)\right) \quad (51)$$

Evaluating (45) from (37), (42) and (43) we group terms into two parts $h_\beta = h_{\beta_1} + h_{\beta_2}$ with

$$\begin{aligned} h_{\beta_1} &= r^{(3)} - \gamma_{\theta_1}\Gamma_v\widetilde{v} - \gamma_{\theta_1}\alpha\left(\hat{\theta}_1, \delta_a + k_\theta, \tau_2\right)\widetilde{v}\\ &\quad -K_p\phi(\widetilde{p}) + k_v\phi(\widetilde{v})K_p z(\widetilde{p}) & (52)\\ h_{\beta_2} &= W_1 z(\widetilde{v}) + W_2\widetilde{q} & (53)\\ W_1 &= k_v^2\phi(\widetilde{v}) & (54)\\ W_2 &= -2k_v u_t\phi(\widetilde{v})R^TS(\bar{q})\\ &\quad -\gamma_{\theta_1}\gamma_q\phi(\widetilde{v})K_v\Phi(\mu_d)^T & (55) \end{aligned}$$

where the function h_{β_2} is chosen since it is bounded functions of $\widetilde{v}$ and $\widetilde{q}$, and do not necessarily require cancelation using the virtual control. We now define the virtual control input

$$\Omega_d = \Phi(\mu_d)h_{\beta_1} + \frac{2u_t}{\gamma_q}S(\bar{q})R\Gamma_v\widetilde{v} - K_q\widetilde{q} \quad (56)$$

where $K_q = K_q^T > 0$ which results in

$$\begin{aligned} \dot{V}_1 &= -\frac{k_\theta}{\gamma_{\theta_1}}\widetilde{\theta}_1^T\phi(\widetilde{v})\widetilde{\theta}_1 - k_v\widetilde{v}^T\Gamma_v z(\widetilde{v}) + \gamma_q\widetilde{q}^T\widetilde{\Omega}\\ &\quad -\gamma_q\widetilde{q}^T\Phi(\mu_d)W_1 z(\widetilde{v}) - \gamma_q\widetilde{q}^T\left(K_q + \Phi(\mu_d)W_2\right)\widetilde{q}\\ &\quad -\widetilde{\theta}_1^T\alpha\left(\hat{\theta}_1, \delta_a + k_\theta, \tau_2\right) & (57) \end{aligned}$$

Note that the term in (57) involving α is non-positive due to the properties of the smooth projection. To study the bound of $\Phi(\mu_d)$, W_1 and W_2 we first recall from (35) the control law for μ_d. Due to the use of projection, using (22) we can guarantee that $\|\hat{\theta}_1\| < \delta_a + k_\theta$. Therefore, due to assumption 1, μ_d is bounded such that $\|\mu_d\| \le \bar{\mu}_d$ where

$$\bar{\mu}_d \le \delta_r + \delta_a + \|K_p\| + k_v + 2k_\theta + \epsilon_\alpha \quad (58)$$

where $\epsilon_\alpha > 0$ is a gain used in the smooth projection function. Note the third component of μ_d is bounded by $|\mu_{d3}| \le \bar{\mu}_{d3}$ where

$$\bar{\mu}_{d3} = k_1\|\hat{z}^T\Gamma_v^{-1}\| + k_v + 2k_\theta + \delta_a + \delta_{rz} + \epsilon_\alpha \quad (59)$$

Due to the constraint (13) we require that $\bar{\mu}_{d3} < g$, therefore we define $\delta_{\mu_d} = g - \bar{\mu}_{d3} > 0$. From (14), (35), (58) and (59) we find the thrust is bounded by $\delta_{\mu_d} \le u_t \le \bar{u}_t$ where $\bar{u}_t = g + \bar{\mu}_d$. Calculating the Frobenius norm of the matrix given by (19) we find

$$\|\Phi(\mu_d)\|_F = \sqrt{\frac{2}{u_t\left(u_t + |\mu_{d3} - g|\right)} + \frac{1}{u_t^2}} \quad (60)$$

which we use to find the bound $\|\Phi(\mu_d)\|_F \le \sqrt{2}\delta_{\mu_d}^{-1}$. Using (60) and Young's Inequality, we now study the bound of the term involving W_1 in (57), or

$$|\gamma_q\widetilde{q}^T\Phi(\mu_d)W_1z(\widetilde{v})| \le \frac{\gamma_q k_v^2}{2\epsilon_q}\widetilde{q}^T\widetilde{q} + \frac{\gamma_q k_v^2\epsilon_q}{\delta_{\mu_d}^2}\widetilde{v}^Tz(\widetilde{v}) \quad (61)$$

where $\epsilon_q > 0$. From (57) and (61), we define

$$\begin{aligned} \Delta_v &= k_v\Gamma_v - \frac{\gamma_q k_v^2\epsilon_q}{\delta_{\mu_d}^2}I & (62)\\ \Delta_q &= K_q - \left(\frac{2\sqrt{2}k_v\bar{u}_t}{\delta_{\mu_d}} + \frac{2\gamma_{\theta_1}\gamma_q}{\delta_{\mu_d}^2}\|K_v\| + \frac{k_v^2}{2\epsilon_q}\right)I & (63) \end{aligned}$$

where we choose the control gains k_v, Γ_v and K_q such that Δ_v and Δ_q are positive definite, which guarantees that the indefinite terms are always dominated. Therefore, (57) becomes

$$\dot{V}_1 \le -\frac{k_\theta}{\gamma_{\theta_1}}\widetilde{\theta}_1^T\phi(\widetilde{v})\widetilde{\theta}_1 - \widetilde{v}^T\Delta_v z(\widetilde{v}) - \gamma_q\widetilde{q}^T\Delta_q\widetilde{q} + \gamma_q\widetilde{q}^T\widetilde{\Omega} \quad (64)$$

In order to proceed with determining the control torque u, we differentiate (56) obtaining

$$\dot{\Omega}_d = f_{\Omega_d} + g_{\Omega_d}\theta_a \quad (65)$$

where the exact expressions for f_{Ω_d} and g_{Ω_d} are given in the appendix. As shown by (65) we must still deal with the unknown parameter θ_a. Since we have already defined our first adaptation law, we must use over-parameterization of the unknown parameter, and now define a new estimate $\hat{\theta}_2$ and the error $\widetilde{\theta}_2 = \theta_a - \hat{\theta}_2$. Also, we now denote $\hat{\theta}_3$ as the adaptive estimate of the unknown parameter θ_b and define the error $\widetilde{\theta}_3 = \theta_b - \hat{\theta}_3$. We now consider the second Lyapunov candidate

$$V_2 = V_1 + \frac{1}{2}\widetilde{\Omega}^T I_b \widetilde{\Omega} + \frac{1}{2\gamma_{\theta_2}}\widetilde{\theta}_2^T\widetilde{\theta}_2 + \frac{1}{2\gamma_{\theta_3}}\widetilde{\theta}_3^T\widetilde{\theta}_3 \quad (66)$$

where $\gamma_{\theta_2} > 0$, $\gamma_{\theta_3} > 0$, which we differentiate to obtain

$$\begin{aligned}\dot{V}_2 \leq & -\frac{k_\theta}{\gamma_{\theta_1}}\widetilde{\theta}_1^T\phi(\widetilde{v})\widetilde{\theta}_1 - v^T\Delta_v z(\widetilde{v}) - \gamma_q\widetilde{q}^T\Delta_q\widetilde{q} \\ & +\widetilde{\Omega}^T\Big(\gamma_q\widetilde{q} - S(\Omega)I_b\Omega + S(\hat{z})R\hat{\theta}_3 + u \\ & -I_b f_{\Omega_d} - I_b g_{\Omega_d}\hat{\theta}_2\Big) + \widetilde{\theta}_2^T\left(-g_{\Omega_d}^T I_b\widetilde{\Omega} - \frac{1}{\gamma_{\theta_2}}\dot{\hat{\theta}}_2\right) \\ & +\widetilde{\theta}_3^T\left(-R^T S(\hat{z})\widetilde{\Omega} - \frac{1}{\gamma_{\theta_3}}\dot{\hat{\theta}}_3\right) \quad (67)\end{aligned}$$

From (67) we define the control and estimation laws

$$\begin{aligned}u =& -\gamma_q\widetilde{q} + S(\Omega)I_b\Omega - S(\hat{z})R\hat{\theta}_3 + I_b f_{\Omega_d} \\ & +I_b g_{\Omega_d}\hat{\theta}_2 - K_w\widetilde{\Omega} \quad (68)\\ \dot{\hat{\theta}}_2 =& \gamma_{\theta_2}\left(-g_{\Omega_d}^T I_b\widetilde{\Omega} + \alpha\left(\hat{\theta}_2, \delta_a, -g_{\Omega_d}^T I_b\widetilde{\Omega}\right)\right) \quad (69)\\ \dot{\hat{\theta}}_3 =& \gamma_{\theta_3}\left(-R^T S(\hat{z})\widetilde{\Omega} + \alpha\left(\hat{\theta}_3, \delta_b, -R^T S(\hat{z})\widetilde{\Omega}\right)\right) \quad (70)\end{aligned}$$

Note that the use of projection is not required for the estimation laws (69)-(70), yet it is used to improve robustness. Applying (68)-(70) to (67) gives the negative semi-definite result

$$\begin{aligned}\dot{V}_2 \leq & -\frac{k_\theta}{\gamma_{\theta_1}}\widetilde{\theta}_1^T\phi(\widetilde{v})\widetilde{\theta}_1 - \widetilde{v}^T\Delta_v z(\widetilde{v}) \\ & -\gamma_q\widetilde{q}^T\Delta_q\widetilde{q} - \widetilde{\Omega}^T K_w\widetilde{\Omega} \quad (71)\end{aligned}$$

from which we conclude that $\widetilde{p}$, $\widetilde{v}$, $\widetilde{\Omega}$ are bounded. The adaptive estimates were bounded a priori due to the use of projection in the adaptive estimation laws, and the signal $\widetilde{q}$ is bounded by convention. It follows from Barbalats lemma that provided the second, third and fourth derivatives of the reference trajectory $r(t)$ are uniformly continuous, then the signals $\widetilde{v}$, $\widetilde{\theta}_1$, $\widetilde{q}$ and $\widetilde{\Omega}$ converge asymptotically to zero. Also, since we have $\dot{\widetilde{v}} \to 0$ then $\widetilde{p} \to 0$, which satisfies the trajectory tracking objective.

If we consider (41) and (51) given by

$$\begin{pmatrix}\dot{\mu}_d \\ \dot{\hat{\theta}}_1\end{pmatrix} = \begin{pmatrix} f_{\mu_d} + g_{\mu_d}\theta_a \\ \gamma_{\theta_1}\left(\tau_2 + \alpha\left(\hat{\theta}_1, \delta_a + k_\theta, \tau_2\right)\right)\end{pmatrix} \quad (72)$$

and note that τ_2 depends on μ_d, and f_{μ_d} depends on $\hat{\theta}_1$, it becomes apparent that the defined control is dynamic in nature due to the interaction of μ_d and $\hat{\theta}_1$. However, since we know that α is Lipschitz continuous, then the RHS of (72) is also Lipschitz continuous, and thus a solution for the control in terms of μ_d and $\hat{\theta}_1$ always exists.

Theorem 1: Consider the system given by (3)-(6) and the reference trajectory $r(t)$ where assumptions 1 and 2 hold. Using (35), if we specify the desired attitude, Q_d, using (15) and (16), and specify the desired angular velocity, Ω_d, and system thrust, u_t, using (56) and (14), and apply the control torque (68) in addition to the adaptation laws (51), (69) and (70), then the state variables $\tilde{p}(t)$, $\tilde{v}(t)$, $\tilde{q}(t)$, $\tilde{\Omega}(t)$ are globally bounded and converge asymptotically to zero.

Proof: The proof is given above as a direct result of the constructive control design procedure.

VI. CONCLUSION

An adaptive position tracking controller achieving global asymptotic stability, has been proposed for a VTOL-UAV in the presence of external disturbances. The proposed control scheme relies on the use of a quaternion extraction method allowing the extraction of the desired system attitude and thrust from the required force achieving the tracking objective. The quaternion extraction method provides almost global results, except for a mild singularity that can be easily avoided. To facilitate the bounded control, projection is used in the adaptive estimation algorithms. Although, this improves the robustness of the proposed controller, this method requires the use of over-parameterization of the unknown disturbances.

APPENDIX

Recalling (30), we differentiate to obtain $f_\phi(u,v) = \partial/\partial u\,(\phi(u)v)$ where

$$\begin{aligned}f_\phi(u,v) =& \left(1+u^T u\right)^{-5/2}\left[3\left(S(u)^2 - I\right)vu^T\right. \\ & \left.+\left(1+u^T u\right)\left(2S(u)S(v) - S(v)S(u)\right)\right] \quad (73)\end{aligned}$$

Note the function $\Phi(\mu_d)$ can be expressed as $\Phi(\mu_d) = \gamma_\Phi\Psi(\mu_d)$ where $\gamma_\Phi = \left(u_t^2 c_1\right)^{-1}$ and $\Psi(\mu_d)$ is the matrix component of $\Phi(\mu_d)$. We find $\dot{u}_t = \alpha_1^T\dot{\mu}_d$ where $\alpha_1 = u_t^{-1}\left(\mu_d - g\hat{z}\right)$. We define $c_1 = u_t + g - \mu_{d3}$ and note that $\dot{c}_1 = \alpha_2(\mu_d)^T\dot{\mu}_d$ where $\alpha_2(\mu_d) = \alpha_1(\mu_d) - \hat{z}$, then

$$\begin{aligned}\tfrac{\partial}{\partial\mu_d}\Phi(\mu_d)v =& Z_1(\mu_d, v) \\ =& \Psi(\mu_d)vM_\gamma + \gamma_\Phi\Lambda_1(\mu_d, v) \\ M_\gamma =& \gamma_\Phi^2\left(g\hat{z} - \mu_d\right)^T\left(3c_1 I_{3\times3} + S(\hat{z})S(g\hat{z} - \mu_d)\right)\end{aligned}$$

$$\begin{aligned}\Lambda_1 =& M_1 \\ & +\begin{pmatrix} c_1 v_2 & -c_1 v_1 & \mu_{d2}v_1 - \mu_{d1}v_2\end{pmatrix}^T\alpha_1(\mu_d)^T \\ & +\begin{pmatrix} u_t v_2 + \mu_{d2}v_3 & -u_t v_1 - \mu_{d1}v_3 & 0\end{pmatrix}^T\alpha_2(\mu_d)^T \\ M_1 =& \begin{pmatrix} M_{1A} & M_{1B} & \mathbf{0}_{3\times1}\end{pmatrix} \\ M_{1A} =& \begin{pmatrix} -\mu_{d2}v_1 & 2v_1\mu_{d1} + v_2\mu_{d2} - c_1 v_3 & -u_t v_2\end{pmatrix}^T \\ M_{1B} =& \begin{pmatrix} -\mu_{d1}v_1 - 2v_2\mu_{d2} + c_1 v_3 & \mu_{d1}v_2 & u_t v_1\end{pmatrix}^T\end{aligned}$$

Similarly, one can find

$$\begin{aligned}\frac{\partial}{\partial \mu_d}\Phi(\mu_d)^T v &= Z_2(\mu_d, v) \quad (74)\\ Z_2(\mu_d, v) &= \Psi(\mu_d)^T v M_\gamma + \gamma_\Phi \Lambda_2(\mu_d, v)\end{aligned}$$

$$\begin{aligned}\Lambda_2 &= M_2\\ &+ \begin{pmatrix} \mu_{d2}v_3 - c_1 v_2 & c_1 v_1 - \mu_{d1} v_3 & 0 \end{pmatrix}^T \alpha_1(\mu_d)^T\\ &+ \begin{pmatrix} -u_t v_2 & u_t v_1 & \mu_{d2}v_1 - \mu_{d1}v_2 \end{pmatrix}^T \alpha_2(\mu_d)^T\\ M_2 &= \begin{pmatrix} M_{2A} & M_{2B} & \mathbf{0}_{3\times 1} \end{pmatrix}\\ M_{2A} &= \begin{pmatrix} 2v_2\mu_{d1} - \mu_{d2}v_1 & \mu_{d2}v_2 - u_t v_3 & -c_1 v_2 \end{pmatrix}^T\\ M_{2B} &= \begin{pmatrix} u_t v_3 - \mu_{d1}v_1 & \mu_{d1}v_2 - 2\mu_{d2}v_1 & c_1 v_1 \end{pmatrix}^T\end{aligned}$$

Recall (47) and denote $\dot{\tilde{q}} = f_{\tilde{q}} + g_{\tilde{q}}\theta_a$, where

$$\begin{aligned}f_{\tilde{q}} &= \frac{1}{2}\left(\tilde{q}_0 I + S(\tilde{q})\right)\Omega\\ &+ \frac{1}{2}\left(S(\tilde{q}) - \tilde{q}_0 I\right)\left(\Phi(\mu_d) f_{\mu_d}\right) \quad (75)\\ g_{\tilde{q}} &= \frac{1}{2}\left(S(\tilde{q}) - \tilde{q}_0 I\right)\left(\Phi(\mu_d) g_{\mu_d}\right) \quad (76)\end{aligned}$$

Recall (10) we find $\dot{\tilde{\mu}} = f_{\tilde{\mu}} + g_{\tilde{\mu}}\theta_a$ where

$$f_{\tilde{\mu}} = -\left(I + u_t^{-1} R^T \hat{z}\left(\mu_d - g\hat{z}\right)^T\right) f_{\mu_d} - u_t R^T S(\Omega)\hat{z} \quad (77)$$

$$g_{\tilde{\mu}} = -\left(I + u_t^{-1} R^T \hat{z}\left(\mu_d - g\hat{z}\right)^T\right) g_{\mu_d} \quad (78)$$

Recall (32), from which we find $\dot{f}_{\tilde{v}_1} = f_{f_{\tilde{v}_1}} + g_{f_{\tilde{v}_1}}\theta_a$, where

$$\begin{aligned}f_{f_{\tilde{v}_1}} &= -K_p\phi(\tilde{p})\tilde{v} - k_v\phi(\tilde{v}) f_{\tilde{v}_2} + f_{\tilde{\mu}} \quad (79)\\ g_{f_{\tilde{v}_1}} &= -k_v\phi(\tilde{v}) + g_{\tilde{\mu}} \quad (80)\end{aligned}$$

From (50) we find $\dot{\tau}_2 = f_{\tau_2} + g_{\tau_2}\theta_a$ where

$$\begin{aligned}f_{\tau_2} &= \Gamma_v f_{\tilde{v}_2} - \frac{k_\theta}{\gamma_{\theta_1}}\left(f_\phi(\tilde{v}, f_{\tilde{v}_1}) f_{\tilde{v}_2} - \phi(\tilde{v})K_p\phi(\tilde{p})\tilde{v}\right.\\ &\left. - k_v\phi(\tilde{v})^2 f_{\tilde{v}_2} + \phi(\tilde{v}) f_{\tilde{\mu}}\right)\\ &+ \gamma_q\left(f_\phi(\tilde{v}, K_v\Phi(\mu_d)^T\tilde{q}) f_{\tilde{v}_2} + \phi(\tilde{v})K_v Z_2(\mu_d, \tilde{q}) f_{\mu_d}\right.\\ &\left. + \phi(\tilde{v})K_v\Phi(\mu_d)^T f_{\tilde{q}}\right) \quad (81)\\ g_{\tau_2} &= \Gamma_v - \frac{k_\theta}{\gamma_{\theta_1}}\left(f_\phi(\tilde{v}, f_{\tilde{v}_1}) - k_v^2\phi(\tilde{v})^2 + \phi(\tilde{v}) g_{\tilde{\mu}}\right)\\ &+ \gamma_q\left(f_\phi(\tilde{v}, K_v\Phi(\mu_d)^T\tilde{q}) + \phi(\tilde{v})K_v Z_2(\mu_d, \tilde{q}) g_{\mu_d}\right.\\ &\left. + \phi(\tilde{v})K_v\Phi(\mu_d)^T g_{\tilde{q}}\right) \quad (82)\end{aligned}$$

In light of the work presented in [12], we differentiate the smooth projection. We define $\theta_0 = \delta_a + k_\theta$ and $\dot{\alpha}(\hat{\theta}_1, \theta_0, \tau_2) = f_\alpha + g_\alpha\theta_a$ where

$$\begin{aligned}f_\alpha &= -k_\alpha\dot{\eta}_1\eta_2\hat{\theta}_1 - k_\alpha\eta_1\eta_2\dot{\hat{\theta}}_1\\ &- k_\alpha\eta_1\frac{\eta_2}{\eta_2 - \hat{\theta}_1\tau_2}\left(\tau_2^T\dot{\hat{\theta}}_1 + \hat{\theta}_1^T f_{\tau_2}\right)\hat{\theta}_1 \quad (83)\\ g_\alpha &= -k_\alpha\eta_1\frac{\eta_2}{\eta_2 - \hat{\theta}_1^T\tau_2}\hat{\theta}_1\hat{\theta}_1^T g_{\tau_2} \quad (84)\end{aligned}$$

$$\dot{\eta}_1 = \begin{cases} 4\left(\hat{\theta}_1^T\hat{\theta}_1 - \theta_0^2\right)\hat{\theta}_1^T\dot{\hat{\theta}}_1 & \text{if} \quad \|\hat{\theta}_1\|^2 > \theta_0^2\\ 0 & \text{otherwise}\end{cases} \quad (85)$$

where $\theta_0 = \delta_p + k_\theta$. We also find $\dot{\bar{q}} = f_{\bar{q}} + g_{\bar{q}}\theta_a$

$$\begin{aligned}f_{\bar{q}} &= S(\hat{z}) f_{\tilde{q}} + \frac{1}{2}\hat{z}\tilde{q}^T\Phi(\mu_d) f_{\mu_d} - \frac{1}{2}\hat{z}\tilde{q}^T\Omega \quad (86)\\ g_{\bar{q}} &= S(\hat{z}) g_{\tilde{q}} + \frac{1}{2}\hat{z}\tilde{q}^T\Phi(\mu_d) g_{\mu_d} \quad (87)\end{aligned}$$

From (52) we differentiate to obtain $\dot{h}_{\beta_1} = f_{h_{\beta_1}} + g_{h_{\beta_1}}\theta_a$ with

$$\begin{aligned}f_{h_{\beta_1}} &= r^{(4)} - \gamma_{\theta_1}\Gamma_v f_{\tilde{v}_2} - \gamma_{\theta_1} f_\alpha - K_p f_\phi(\tilde{p}, \tilde{v})\tilde{v}\\ &- K_p\phi(\tilde{p}) f_{\tilde{v}_2} + k_v f_\phi(\tilde{v}, K_p z(\tilde{p})) f_{\tilde{v}_2}\\ &+ k_v\phi(\tilde{v})K_p\phi(\tilde{p})\tilde{v} \quad (88)\\ g_{h_{\beta_1}} &= -\gamma_{\theta_1}\Gamma_v - \gamma_{\theta_1} g_\alpha - K_p\phi(\tilde{p})\\ &+ k_v f_\phi(\tilde{v}, K_p z(\tilde{p})) \quad (89)\end{aligned}$$

Finally, in light of the above results we can find

$$\begin{aligned}f_{\Omega_d} &= Z_1(\mu_d, f_{\beta_1}) f_{\mu_d} + \Phi(\mu_d) f_{h_{\beta_1}} - K_q f_{\tilde{q}}\\ &+ \frac{2}{\gamma_q}\left[u_t^{-1} S(\bar{q}) R\Gamma_v\tilde{v}\left(\mu_d - g\hat{z}\right)^T f_{\mu_d}\right.\\ &- u_t S(R\Gamma_v\tilde{v}) f_{\bar{q}} - u_t S(\bar{q}) S(\Omega) R\Gamma_v\tilde{v}\\ &\left. + u_t S(\bar{q}) R\Gamma_v f_{\tilde{v}_2}\right] \quad (90)\\ g_{\Omega_d} &= Z_1(\mu_d, f_{\beta_1}) g_{\mu_d} + \Phi(\mu_d) g_{h_{\beta_1}} - K_q g_{\tilde{q}}\\ &+ \frac{2}{\gamma_q}\left[u_t^{-1} S(\bar{q}) R\Gamma_v\tilde{v}\left(\mu_d - g\hat{z}\right)^T g_{\mu_d}\right.\\ &\left. - u_t S(R\Gamma_v\tilde{v}) g_{\bar{q}} + u_t S(\bar{q}) R\Gamma_v\right] \quad (91)\end{aligned}$$

REFERENCES

[1] M.-D. Hua, P. Morin, and C. Samson, "Balanced-force-control of underactuated thrust-propelled vehicles," in *Proc. 46th IEEE Conference on Decision and Control*, 12–14 Dec. 2007, pp. 6435–6441.

[2] M.-D. Hua, T. Hamel, P. Morin, and C. Samson, "Control of thrust-propelled underactuated vehicles," *INRIA Technical Report*, 2008.

[3] J. Pflimlin, P. Soueres, and T. Hamel, "Position control of a ducted fan vtol uav in crosswind," *International Journal of Control*, vol. 80, no. 5, pp. 666–683, 2007.

[4] J. Hauser, S. Sastry, and G. Meyer, "Nonlinear control design for slightly non-minimum phase systems: Application to v/stol aircraft," *Automatica*, vol. 28, no. 4, pp. 665–679, 1992.

[5] S. Salazar-Cruz, A. Palomino, and R. Lozano, "Trajectory tracking for a four rotor mini-aircraft," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on*, Dec. 2005, pp. 2505–2510.

[6] R. Olfati-Saber, "Global configuration stabilization for the vtol aircraft with strong input coupling," *IEEE Trans. Autom. Control*, vol. 47, no. 11, pp. 1949–1952, Nov. 2002.

[7] A. P. Aguiar and J. P. Hespanha, "Position tracking of underactuated vehicles," in *Proc. American Control Conference the 2003*, vol. 3, 4–6 June 2003, pp. 1988–1993.

[8] E. Frazzoli, M. A. Dahleh, and E. Feron, "Trajectory tracking control design for autonomous helicopters using a backstepping algorithm," in *Proc. American Control Conference the 2000*, vol. 6, 28–30 June 2000, pp. 4102–4107.

[9] M. Shuster and S. Oh, "Three-axis attitude determination from vector observations," *Journal of Guidance and Control*, vol. 4, pp. 70–77, 1979.

[10] N. Metni, J. M. Pflimlin, T. Hamel, and P. Soueres, "Attitude and gyro bias estimation for a flying uav," in *Proc. IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS 2005)*, 2–6 Aug. 2005, pp. 1114–1120.

[11] I. P. and J. Sun, *Robust Adaptive Control*. Prentice Hall, Inc., 1996.

[12] Z. Cai, M. S. de Queiroz, and D. M. Dawson, "A sufficiently smooth projection operator," *IEEE Trans. Autom. Control*, vol. 51, no. 1, pp. 135–139, Jan. 2006.