Proceedings of the 45th IEEE Conference on Decision & Control
Manchester Grand Hyatt Hotel
San Diego, CA, USA, December 13-15, 2006

An Adaptive Iterative Learning Control Framework for a Class of Uncertain Nonlinear Systems

Abdelhamid Tayebi and Chiang-Ju Chien

Abstract— In this paper, we propose a unified framework for adaptive iterative learning control design for uncertain nonlinear systems. It is shown that if a Lyapunov based adaptive control law is available for the system under consideration and the Lyapunov function satisfies certain conditions, it is straightforward to extend the adaptive controller to handle repetitive systems operating over a finite time interval. According to the value of a certain parameter γ, the parametric adaptation law can be a pure time-domain adaptation, a pure iteration-domain adaptation or a combination of both. The advantages and disadvantages of the three possible adaptation types are discussed and some illustrative examples are given.

I. Introduction

After more that two decades of intensive research, iterative learning control (ILC) is now a well established control techniques that fits well systems that are repetitive in nature. Roughly speaking, this technique aims to generate, in an iterative manner, the adequate control input leading to a 'perfect' tracking over a finite time-interval for systems executing repetitive tasks over a finite time-interval (See, for instance, [1], [2], [3], [4], [11], [13]).

In its early stages, the design of ILC schemes was, primarily, based upon the contraction mapping approach and the use of the time-weighted norm (or λ-norm) to prove the convergence of the iterative process. This approach, basically, consists of adjusting the previous control input with an adequate correcting term depending, generally, on the current and/or the previous tracking error profiles. This approach encountered several well known obstacles such as the resetting condition, the low convergence rates, the requirement of the global Lipschitz condition for nonlinear systems as well as the use of the output time-derivatives for systems with high relative degree. In this framework, the reference trajectory as well as the disturbances are usually assumed to be iteration-invariant (i.e., the reference trajectory (or the disturbance) has to be the same at each iteration). In the mid 90s, a new ILC approach, namely adaptive ILC, based on a Lyapunov-like theory, has been introduced to overcome some of the limitations of the original approach [5], [6], [7], [8], [12], [13], [14]. This new design methodology, which inherits the main attributes from its counterpart in standard nonlinear theory, provided powerful tools to handle complex systems that were difficult to handle using the contraction mapping approach. In fact, among the benefits of this approach, one can recall the relaxation of the resetting and Lipschitz conditions, the ability to handle systems with high relative degree, as well as iteration-varying disturbances and reference trajectories. In this framework, the previous control input is adjusted indirectly through the adjustment of some parameters in the control law. The adjustment of the parameters can be performed along the iteration axis [12], [13], [14], along the time-axis (initializing the parameter estimates with their final values obtained at the preceding iteration) [6], or combining both [7], [8], [10].

In this paper, we provide a unified formulation of adaptive ILC for uncertain nonlinear systems. In fact, we provide a systematic procedure for the design of adaptive ILC schemes for uncertain systems based on the existence of a Lyapunov function for the system under consideration. The proposed parametric adaptation law is quite general in the sense that it depends on a scalar γ allowing to select the desired type among the three adaptation types discussed above, namely, a pure time-domain adaptation for $\gamma = 0$, a pure iteration-domain adaptation for $\gamma = 1$, and a combination of both for $\gamma \in (0, 1)$. In this framework, the reference trajectory is allowed to be iteration-varying and the initial tracking error, at each iteration, is either set to zero (resetting condition) or to the tracking error obtained at the end of the previous iteration (alignment condition). The advantages and disadvantages of the three adaptation types are discussed and some examples illustrating the design procedure are provided.

II. Adaptive ILC design

Let us consider the following nonlinear system

$$\dot{x}_k(t) = f(x_k(t), u_k(t), \theta, t), \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector (denoting generally the tracking error), $u_k \in \mathbb{R}^m$ is the control vector, $\theta \in \mathbb{R}^p$ is an unknown constant vector, $t \in [0, T]$ is the time, and $k \in \mathbb{Z}_+$ in the iteration (or trial) index. The nonlinear function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times [0, T] \to \mathbb{R}^n$ is such that $f(x_k(t), u_k(t), \theta, t)$ is bounded over $[0, T]$

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC), and the National Science Council (NSC), R.O.C.

A. Tayebi is with the Department of Electrical Engineering, Lakehead University, Thunder Bay, Ontario, P7B 5E1, Canada. atayebi@lakeheadu.ca

C-J. Chien is with the Department of Electronic Engineering, Huafan University, 223, Shihtin, Taipei County, Taiwan. cjc@huafan.hfu.edu.tw

as long as $x_k(t)$ and $u_k(t)$ are bounded over $[0,T]$. In general, system (1) represents the error dynamics and the explicit appearance of the time in (1) is often due to the time-varying reference trajectory.

Suppose that one can design a dynamic control law of the form

$$\begin{aligned} u_k(t) &= g(x_k(t), \hat{\theta}_k(t), t) \\ \dot{\hat{\theta}}_k(t) &= h(x_k(t), t), \end{aligned} \quad (2)$$

such that there exists a positive definite function

$$\Phi(x_k, \tilde{\theta}_k) = V(x_k) + W(\tilde{\theta}_k), \quad (3)$$

satisfying

$$\dot{\Phi} = L_f V(x_k) + L_h W(\tilde{\theta}_k) \le -\Upsilon(x_k), \quad (4)$$

where $\Upsilon(x_k)$ is a positive definite function, $\tilde{\theta}_k = \hat{\theta}_k - \theta$, and $L_f V \equiv \frac{\partial V}{\partial x_k} f$ and $L_h W \equiv \frac{\partial W}{\partial \tilde{\theta}_k} h$.

We assume that $h(x_k(t), t)$ is bounded over $[0,T]$ as long as $x_k(t)$ is bounded over $[0,T]$. We also assume that $W(\tilde{\theta})$ satisfies the following properties:

$P1)$

$$\left\| \frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} \right\|^2 \le \frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} \tilde{\theta}_k, \quad (5)$$

$P2)$

$$W(\tilde{\theta}_k) - W(\tilde{\theta}_{k-1}) \le -\Omega(\bar{\theta}_k) + \frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} \bar{\theta}_k \quad (6)$$

where $\Omega(\bar{\theta}_k)$ is a positive semi-definite function, and $\bar{\theta}_k(t) = \hat{\theta}_k(t) - \hat{\theta}_{k-1}(t)$.

Note that the properties P1 and P2 are satisfied if we consider, for instance, $W(\tilde{\theta}_k) = \frac{1}{2}\tilde{\theta}_k^T \Gamma^{-1} \tilde{\theta}_k$, with Γ being a symmetric positive definite matrix. In this particular case, $\frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} = \tilde{\theta}_k^T \Gamma^{-1}$ and $W(\tilde{\theta}_k(t)) - W(\tilde{\theta}_{k-1}(t)) = -\frac{1}{2}\bar{\theta}_k^T \Gamma^{-1} \bar{\theta}_k + \bar{\theta}_k^T \Gamma^{-1} \tilde{\theta}_k$.

Throughout this paper, we will use the $\mathcal{L}_{pe}$ norm defined as follows:

$$\|x(t)\|_{pe} \triangleq \begin{cases} \left(\int_0^t \|x(\tau)\|^p d\tau \right)^{1/p} & if \quad p \in [0, \infty) \\ \sup_{0 \le \tau \le t} \|x(\tau)\| \quad if \quad p = \infty \end{cases}$$

where $\|x\|$ denotes any consistent norm of x, and t belongs to the finite interval $[0,T]$. We say that $x \in \mathcal{L}_{pe}$ when $\|x\|_{pe}$ exists (i.e., when $\|x\|_{pe}$ is finite).

Now, one can state our result in the following theorem

Theorem 1: Consider system (1) under the following adaptive ILC scheme

$$\begin{aligned} u_k(t) &= g(x_k(t), \hat{\theta}_k(t), t) \\ (1-\gamma)\dot{\hat{\theta}}_k(t) &= -\gamma \hat{\theta}_k(t) + \gamma \hat{\theta}_{k-1}(t) + h(x_k(t), t), \end{aligned} \quad (7)$$

with $\gamma \in [0,1]$, $\hat{\theta}_{-1}(t) = 0$. For $\gamma \in [0,1)$, we set $\hat{\theta}_k(0) = \hat{\theta}_{k-1}(T)$. Assume that $x_k(0) = 0$ or $x_k(0) = x_{k-1}(T)$. Then

i) For $\gamma \in [0,1)$, $x_k(t), \tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{\infty e}$, for all $k \in \mathbb{Z}_+$ and for all $t \in [0,T]$, and $\lim_{k\to\infty} x_k(t) = 0$, $\forall t \in [0,T]$.

ii) For $\gamma = 1$, $x_k(t) \in \mathcal{L}_{\infty e}$, $\tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{2e}$ for all $k \in \mathbb{Z}_+$ and for all $t \in [0,T]$, and $\lim_{k\to\infty} \int_0^T \Upsilon(x_k(\tau)) d\tau = 0$.

Proof: First, we will prove (i), i.e., for $\gamma \in [0,1)$. Let us consider the following positive definite function

$$\Psi(x_k, \tilde{\theta}_k) = V(x_k) + (1-\gamma) W(\tilde{\theta}_k). \quad (8)$$

In the sequel, we will use $\Psi_k(t)$ to denote $\Psi(x_k(t), \tilde{\theta}_k(t))$, $V_k(t)$ to denote $V(x_k(t))$ and $W_k(t)$ to denote $W(\tilde{\theta}_k(t))$. The time derivative of (8), in view of (1-5), is given by

$$\begin{aligned} \dot{\Psi}_k &= L_f V_k + (1-\gamma)\frac{\partial W_k}{\partial \tilde{\theta}_k}\dot{\hat{\theta}}_k, \\ &= L_f V_k + L_h W_k + \frac{\partial W_k}{\partial \tilde{\theta}_k}(-\gamma\hat{\theta}_k + \gamma\hat{\theta}_{k-1}) \\ &\le -\Upsilon(x_k) + \frac{\partial W_k}{\partial \tilde{\theta}_k}(-\gamma\hat{\theta}_k + \gamma\hat{\theta}_{k-1}) \\ &\le -\gamma\frac{\partial W_k}{\partial \tilde{\theta}_k}(\hat{\theta}_k - \hat{\theta}_{k-1}) = -\gamma\frac{\partial W_k}{\partial \tilde{\theta}_k}(\tilde{\theta}_k - \tilde{\theta}_{k-1}) \\ &\le -\gamma\frac{\partial W_k}{\partial \tilde{\theta}_k}\tilde{\theta}_k + \gamma\kappa\|\frac{\partial W_k}{\partial \tilde{\theta}_k}\|^2 + \frac{\gamma}{4\kappa}\|\tilde{\theta}_{k-1}\|^2 \\ &\le \frac{\gamma}{4\kappa}\|\tilde{\theta}_{k-1}\|^2, \end{aligned} \quad (9)$$

for any $0 < \kappa \le 1$. Since $\hat{\theta}_{-1}(t) = 0$ and $\hat{\theta}_0(0) = \hat{\theta}_{-1}(T)$, it is clear that $\Psi_0(t)$ and hence $x_0(t)$ and $\tilde{\theta}_0(t)$ are bounded for all $t \in [0,T]$.

Now, let us use the following positive definite functional

$$\bar{\Psi}(x_k, \tilde{\theta}_k, t) = \Psi(x_k, \tilde{\theta}_k) + \gamma \int_0^t W(\tilde{\theta}_k(\tau)) d\tau, \quad (10)$$

whose difference can be evaluated, in view of (1-6), as follows:

$$\begin{aligned} \Delta\bar{\Psi}_k(t) &= \Psi_k(t) - \Psi_{k-1}(t) \\ &\quad +\gamma \int_0^t (W_k(\tau) - W_{k-1}(\tau)) d\tau \\ &= V_k(t) - V_{k-1}(t) \\ &\quad +(1-\gamma)(W_k(t) - W_{k-1}(t)) \\ &\quad +\gamma \int_0^t (W_k(\tau) - W_{k-1}(\tau)) d\tau \\ &= -V_{k-1}(t) - (1-\gamma)W_{k-1}(t) + V_k(0) \\ &\quad +(1-\gamma)W_k(0) \\ &\quad +\gamma \int_0^t (W_k(\tau) - W_{k-1}(\tau)) d\tau \\ &\quad + \int_0^t (L_f V_k(\tau) + (1-\gamma)\frac{\partial W_k}{\partial \tilde{\theta}_k}\dot{\hat{\theta}}_k(\tau)) d\tau \\ &\le -\int_0^t \Upsilon(x_k(\tau)) d\tau \\ &\quad -\gamma \int_0^t \Omega(\bar{\theta}_k) d\tau - V_{k-1}(t) \\ &\quad -(1-\gamma)W_{k-1}(t) + V_k(0) + (1-\gamma)W_k(0), \end{aligned} \quad (11)$$

Now, using the fact that $V_k(0) = 0$ (or $V_k(0) = V_{k-1}(T)$) and $W_k(0) = W_{k-1}(T)$, we have,

$$\Delta\bar{\Psi}_k(T) \le -\int_0^T \Upsilon(x_k(\tau))d\tau - \gamma\int_0^T \Omega(\bar{\theta}_k(\tau))d\tau \le 0. \tag{12}$$

Hence, $\bar{\Psi}_k(T)$ is bounded for all $k \in \mathbb{Z}_+$ since $\bar{\Psi}_0(T)$ is bounded due to the boundedness of $\Psi_0(t)$ over $[0,T]$. This implies that $\Psi_k(T)$ and $\int_0^T W(\tilde{\theta}_k(\tau))d\tau$ are bounded for all $k \in \mathbb{Z}_+$, which in turn implies that $\int_0^T \|\tilde{\theta}_k\|^2 d\tau$ is bounded for all $k \in \mathbb{Z}_+$. Now, from (9), in the case where $V_k(0) = V_{k-1}(T)$, one has

$$\begin{aligned}\Psi_k(t) &\le \Psi_k(0) + \int_0^t \frac{\gamma}{4\kappa}\|\tilde{\theta}_{k-1}\|^2 d\tau,\\ &\le \Psi_{k-1}(T) + \int_0^T \frac{\gamma}{4\kappa}\|\tilde{\theta}_{k-1}\|^2 d\tau,\end{aligned}$$

and in the case where $V_k(0) = 0$, one has

$$\Psi_k(t) \le (1-\gamma)W(\tilde{\theta}_{k-1}(T)) + \int_0^T \frac{\gamma}{4\kappa}\|\tilde{\theta}_{k-1}\|^2 d\tau,$$

which implies that $\Psi_k(t)$ is bounded for all $k \in \mathbb{Z}_+$ and all $t \in [0,T]$. Now, from (12), it is easily seen that

$$\begin{aligned}\bar{\Psi}_k(T) &= \bar{\Psi}_k(0) + \sum_{j=1}^{k}\Delta\bar{\Psi}_j(T)\\ &\le \bar{\Psi}_k(0) - \sum_{j=1}^{k}\int_0^T \Upsilon(x_j(\tau))d\tau \\ &\quad -\gamma\sum_{j=1}^{k}\int_0^T \Omega(\bar{\theta}_j(\tau))d\tau.\end{aligned} \tag{13}$$

Hence,

$$\begin{aligned}&\sum_{j=1}^{k}\int_0^T \Upsilon(x_j(\tau))d\tau\\ &+\gamma\sum_{j=1}^{k}\int_0^T \Omega(\bar{\theta}_j(\tau))d\tau \le \bar{\Psi}_k(0) - \bar{\Psi}_k(T).\end{aligned} \tag{14}$$

Since $\Psi_k(t)$ is bounded for all $k \in \mathbb{Z}_+$, and for all $t \in [0,T]$, it is clear that $\bar{\Psi}_k(t)$ is bounded for all $k \in \mathbb{Z}_+$, and for all $t \in [0,T]$. Therefore, from (14), one can conclude that $\lim_{k\to\infty}\int_0^T \Upsilon(x_k(\tau))d\tau = 0$ and $\lim_{k\to\infty}\int_0^T \Omega(\bar{\theta}_k(\tau))d\tau = 0$. Since $x_k(t), \tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{\infty e}$, one has $\dot{x}_k(t) \in \mathcal{L}_{\infty e}$. Consequently, one can conclude that $\lim_{k\to\infty}\Upsilon(x_k(t)) = 0$ for all $t \in [0,T]$ and hence $\lim_{k\to\infty} x_k(t) = 0$ for all $t \in [0,T]$.

Now, let us prove (ii), i.e., for $\gamma = 1$. Consider the following positive definite functional

$$\bar{\Psi}(x_k, \tilde{\theta}_k, t) = V_k(t) + \int_0^t W(\tilde{\theta}_k(\tau))d\tau, \tag{15}$$

whose time derivative, in view of (1-5) and (6), is given by

$$\begin{aligned}\dot{\bar{\Psi}}_k(t) &= \dot{V}_k(t) + W_k(t)\\ &= L_f V_k + W_k(t) - W_{k-1}(t) + W_{k-1}(t)\\ &\le L_f V_k - \Omega(\bar{\theta}_k) + \frac{\partial W(\tilde{\theta}_k)}{\partial\tilde{\theta}_k}\bar{\theta}_k + W_{k-1}(t)\\ &= L_f V_k + L_h W_k - \Omega(\bar{\theta}_k) + W_{k-1}(t)\\ &\le -\Upsilon(x_k) - \Omega(\bar{\theta}_k) + W_{k-1}(t)\\ &\le W_{k-1}(t).\end{aligned} \tag{16}$$

Since $\hat{\theta}_{-1}(t) = 0$, it is clear that $\hat{\theta}_0(t) = h(x_0(t), t)$. Since $x_0(0)$ is bounded, it is clear that $\hat{\theta}_0(0)$ is bounded. Therefore, from (16), it is clear that $\bar{\Psi}_0(t)$ is bounded for all $t \in [0,T]$. The difference of $\bar{\Psi}_k(t)$ can be evaluated, in view of (1-4) and (6), as follows:

$$\begin{aligned}\Delta\bar{\Psi}_k(t) &= V_k(t) - V_{k-1}(t)\\ &\quad + \int_0^t (W_k(\tau) - W_{k-1}(\tau))d\tau\\ &= -V_{k-1}(t) + V_k(0) + \int_0^t L_f V_k(\tau)d\tau\\ &\quad + \int_0^t (W_k(\tau) - W_{k-1}(\tau))d\tau\\ &\le -\int_0^t \Upsilon(x_k(\tau))d\tau - \int_0^t \Omega(\bar{\theta}_k)d\tau\\ &\quad -V_{k-1}(t) + V_k(0),\end{aligned} \tag{17}$$

Now, using the fact that $V_k(0) = 0$ (or $V_k(0) = V_{k-1}(T)$), we have,

$$\Delta\bar{\Psi}_k(T) \le -\int_0^T \Upsilon(x_k(\tau))d\tau - \int_0^T \Omega(\bar{\theta}_k(\tau))d\tau \le 0, \tag{18}$$

which implies that $\bar{\Psi}_k(T)$ is bounded for all $k \in \mathbb{Z}_+$ since $\bar{\Psi}_0(T)$ is bounded.

Let $\varpi_k(t) = \int_0^t W(\tilde{\theta}_k(\tau))d\tau$. It clear that $\varpi_k(t) \le \varpi_k(T) \le \varpi < \infty$ for all $t \in [0,T]$. Therefore

$$\bar{\Psi}_k(t) = V_k(t) + \varpi_k(t) \le V_k(t) + \varpi. \tag{19}$$

Thus,

$$\bar{\Psi}_{k-1}(t) \le V_{k-1}(t) + \varpi. \tag{20}$$

On the other hand, one has

$$\begin{aligned}\Delta\bar{\Psi}_k(t) &= \bar{\Psi}_k(t) - \bar{\Psi}_{k-1}(t)\\ &\le -\int_0^t \Upsilon(x_k(\tau))d\tau - \int_0^t \Omega(\bar{\theta}_k)d\tau\\ &\quad -V_{k-1}(t) + V_k(0)\\ &\le V_k(0) - V_{k-1}(t).\end{aligned} \tag{21}$$

From (20) and (21), one can conclude that

$$\bar{\Psi}_k(t) \le V_k(0) + \varpi \tag{22}$$

- Case 1: $(V_k(0) = V_{k-1}(T))$
  Since $\bar{\Psi}_k(T)$ is bounded $\forall k \in \mathbb{Z}_+$, it is clear that $V_k(T)$ is bounded $\forall k \in \mathbb{Z}_+$. Hence, from (22), one can conclude that $\bar{\Psi}_k(t)$ is bounded $\forall k \in \mathbb{Z}_+$, $\forall t \in [0,T]$.

- Case 2: $(V_k(0) = 0)$
  It is clear that $\bar{\Psi}_k(t)$ is bounded $\forall k \in \mathbb{Z}_+$, $\forall t \in [0, T]$.

Consequently, $x_k(t) \in \mathcal{L}_{\infty e}$, $\tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{2e}$ for all $k \in \mathbb{Z}_+$ and for all $t \in [0, T]$.
We have also

$$\sum_{j=1}^{k} \int_0^T \Upsilon(x_j(\tau))d\tau + \sum_{j=1}^{k} \int_0^T \Omega(\bar{\theta}_j(\tau))d\tau \leq \bar{\Psi}_k(0) - \bar{\Psi}_k(T), \tag{23}$$

which implies that $\lim_{k\to\infty} \int_0^T \Upsilon(x_k(\tau))d\tau = 0$ and $\lim_{k\to\infty} \int_0^T \Omega(\bar{\theta}_k(\tau))d\tau = 0$.

□

Now, we would like to provide the following remarks:
Remark 1: Note that the parametric adaptation law given in (7), is a mixed time-domain and iteration-domain adaptation mechanism for $\gamma \in (0, 1)$. In the case where $\gamma = 0$, the adaptation law becomes a pure time-domain adaptation [6], while for $\gamma = 1$ it becomes a pure iteration-domain adaptation [13]. With $\gamma \in [0, 1)$, we guarantee the boundedness of the infinity norm of the tracking error and the control input as well as the convergence to zero of the infinity norm of the tracking error. With $\gamma = 1$, we guarantee the boundedness of the infinity norm of the tracking error, the boundedness of the $\mathcal{L}_2$-norm of the control input as well as the convergence to zero of the $\mathcal{L}_2$-norm of the tracking error (under the alignment condition) and the convergence to zero of the infinity norm of the tracking error (under the resetting condition)..

Remark 2: With $\gamma = 1$, Property $P1$ is not required to derive the result in Theorem 1, and the unknown vector θ in system (1) can be time-varying. For $\gamma = 0$, both properties $P1$ and $P2$ are not required to derive the result in Theorem 1.

Remark 3: It is worth noting that with $\gamma = 0$, we will need to save only $\hat{\theta}_k(T)$ in the memory instead of saving $\hat{\theta}_k(t),\ t \in [0, T]$. This will considerably contribute to memory space saving in real-time applications.

Remark 4: Using a pure iteration-domain adaptation (i.e., $\gamma = 1$), will avoid the use of the integral to calculate $\hat{\theta}_k$. This is very helpful in real-time applications, since the use of an approximative numerical integration is avoided.

Remark 5: In the case, where $\gamma \in [0, 1)$, h is allowed to depend on $\hat{\theta}_k$, i.e., $h(x_k(t), \hat{\theta}_k(t), t)$. In the case where $\gamma = 1$, one can also allow h to depend on $\hat{\theta}_k$ if we assume that the following adaptation law

$$\hat{\theta}_k(t) = \hat{\theta}_{k-1}(t) + h(x_k(t), \hat{\theta}_k(t), t), \tag{24}$$

has a unique solution $\hat{\theta}_k(t)$, which is bounded over $[0, T]$ if $\hat{\theta}_{k-1}(t)$ and $x_k(t)$ are bounded over $[0, T]$.

Remark 6: It is worth noting that, in the case $\gamma = 1$, one can show that $\lim_{k\to\infty} x_k(t) = 0$, $\forall t \in [0, T]$, as long as $x_k(0) = 0$, $\forall k \in \mathbb{Z}_+$. In fact, in this case, (17) leads to

$$\Delta\bar{\Psi}_k(t) \leq -V_{k-1}(t) - \int_0^t \Upsilon(x_k(\tau))d\tau - \int_0^t \Omega(\bar{\theta}_k(\tau))d\tau \leq 0, \tag{25}$$

which leads to

$$\begin{aligned} &\sum_{j=1}^{k} V_{j-1}(t) + \sum_{j=1}^{k} \int_0^t \Upsilon(x_j(\tau))d\tau \\ &+ \sum_{j=1}^{k} \int_0^t \Omega(\bar{\theta}_j(\tau))d\tau \leq \bar{\Psi}_k(0) - \bar{\Psi}_k(t). \end{aligned} \tag{26}$$

Since $\bar{\Psi}_k(t)$ is bounded $\forall k \in \mathbb{Z}_+$, $\forall t \in [0, T]$, it is clear that $\lim_{k\to\infty} V_k(t) = 0$, which implies that $\lim_{k\to\infty} x_k(t) = 0$, $\forall t \in [0, T]$.

III. Illustrative examples

Example 1:
Consider the following system

$$\dot{x}_k = \theta x_k^2 + u_k, \tag{27}$$

where $x_k \in \mathbb{R}$, $u_k \in \mathbb{R}$, and $\theta \in \mathbb{R}$ is an unknown constant. Assume that the reference trajectory is given by $x_d(t)$. Assume that $x_d(t)$ and $\dot{x}_d(t)$ are bounded over $[0, T]$. The error dynamics is then given by

$$\dot{\tilde{x}}_k = \theta(\tilde{x}_k + x_d(t))^2 + u_k - \dot{x}_d(t), \tag{28}$$

where $\tilde{x}_k = x_k - x_d$. Under the following control law:

$$\begin{aligned} u_k(t) &= -\hat{\theta}_k(t)(\tilde{x}_k + x_d(t))^2 - k\tilde{x}_k + \dot{x}_d(t) \\ \dot{\hat{\theta}}_k(t) &= \beta\tilde{x}_k(\tilde{x}_k + x_d(t))^2, \end{aligned} \tag{29}$$

with $k > 0$, the following positive definite function

$$\Phi(\tilde{x}_k, \tilde{\theta}_k) = \frac{1}{2}\tilde{x}_k^2 + \frac{1}{2\beta}\tilde{\theta}_k^2, \tag{30}$$

satisfies

$$\dot{\Phi} = L_f V(\tilde{x}_k) + L_h W(\tilde{\theta}_k) \leq -k\tilde{x}_k^2. \tag{31}$$

Hence, the adaptive ILC leading to the results in Theorem 1, is given by

$$\begin{aligned} u_k(t) &= -\hat{\theta}_k(t)(\tilde{x}_k + x_d(t))^2 - k\tilde{x}_k + \dot{x}_d(t) \\ (1-\gamma)\dot{\hat{\theta}}_k(t) &= -\gamma\hat{\theta}_k(t) + \gamma\hat{\theta}_{k-1}(t) + \beta\tilde{x}_k(\tilde{x}_k + x_d(t))^2, \end{aligned} \tag{32}$$

with $\beta > 0$, $\gamma \in [0, 1]$ and $\hat{\theta}_{-1}(t) = 0$. For $\gamma \in [0, 1)$, we set $\hat{\theta}_k(0) = \hat{\theta}_{k-1}(T)$. Note that θ is allowed to be time-varying in the case $\gamma = 1$.

Example 2:
Consider the following uncertain nonlinear system:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + x_2 + \theta_1^T\phi_1(x_1) \\ \dot{x}_2 &= f_2(x_1, x_2) + u + \theta_2^T\phi_2(x_1, x_2) \\ y &= x_1, \end{aligned} \tag{33}$$

where f_1 and f_2 are smooth functions, θ_1 and θ_2 are unknown constant vectors of dimension p and q respectively, and ϕ_1 and ϕ_2 are known vectors of smooth functions of dimension p and q respectively. Let $y_d(t)$ be the reference trajectory and $\tilde{x}_1 = x_1 - y_d$ the tracking error. The error dynamics is therefore given by

$$\begin{array}{rcl} \dot{\tilde{x}}_1 &=& \bar{f}_1(\tilde{x}_1,t) + x_2 + \theta_1^T \bar{\phi}_1(\tilde{x}_1,t) \\ \dot{x}_2 &=& \bar{f}_2(\tilde{x}_1,x_2,t) + u + \theta_2^T \bar{\phi}_2(\tilde{x}_1,x_2,t) \end{array} \quad (34)$$

where $\bar{f}_1(\tilde{x}_1,t) = f_1(\tilde{x}_1 + y_d(t)) - \dot{y}_d(t)$, $\bar{f}_2(\tilde{x}_1,x_2,t) = f_2(\tilde{x}_1 + y_d(t), x_2)$, $\bar{\phi}_1(\tilde{x}_1,t) = \phi_1(\tilde{x}_1 + y_d(t))$ and $\bar{\phi}_2(\tilde{x}_1,x_2,t) = \bar{\phi}_2(\tilde{x}_1 + y_d(t), x_2)$.
Applying the backstepping approach for (34), one can design the following adaptive control law:

$$u = -\tilde{x}_1 - \bar{f}_2 - k_2 z_2 - \hat{\theta}_2^T \bar{\phi}_2 + \frac{\partial \psi_1}{\partial \tilde{x}_1}(-k_1\tilde{x}_1 + z_2) + \frac{\partial \psi_1}{\partial t}, \quad (35)$$

with

$$\begin{array}{rcl} \dot{\hat{\theta}}_1 &=& \Gamma_1 \bar{\phi}_1(\tilde{x}_1,t)(\tilde{x}_1 - z_2 \frac{\partial \psi_1}{\partial \tilde{x}_1}) \\ \dot{\hat{\theta}}_2 &=& \Gamma_2 z_2 \bar{\phi}_2(\tilde{x}_1,x_2,t) \end{array} \quad (36)$$

where $k_1, k_2 > 0$, Γ_1, Γ_2 are symmetric positive definite, $\psi_1 = -\bar{f}_1 - \hat{\theta}_1^T \bar{\phi}_1 - k_1 \tilde{x}_1$, $z_2 = x_2 - \psi_1$.
Under this adaptive control law, the following Lyapunov function

$$V(\tilde{x}_1, z_2, \tilde{\theta}_1, \tilde{\theta}_2) = \frac{1}{2}\tilde{x}_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}_1^T\Gamma_1^{-1}\tilde{\theta}_1 + \frac{1}{2}\tilde{\theta}_2^T\Gamma_2^{-1}\tilde{\theta}_2, \quad (37)$$

leads to

$$\dot{V} = -k_1\tilde{x}_1^2 - k_2 z_2^2. \quad (38)$$

This adaptive controller can be extended to the case where system (33) is executing a repetitive task over a finite time-interval. The resulting adaptive ILC will be as in (7), with $\hat{\theta}_k = [\hat{\theta}_{1,k}, \hat{\theta}_{2,k}]^T$,

$$\begin{array}{rcl} g &=& -\tilde{x}_{1,k} - \bar{f}_2(\tilde{x}_{1,k}, x_{2,k}, t) - k_2 z_{2,k} \\ && -\hat{\theta}_{2,k}^T \bar{\phi}_2(\tilde{x}_{1,k}, x_{2,k}, t) \\ && + \frac{\partial \psi_1(\tilde{x}_{1,k},t)}{\partial \tilde{x}_{1,k}}(-k_1\tilde{x}_{1,k} + z_{2,k}) + \frac{\partial \psi_1(\tilde{x}_{1,k},t)}{\partial t} \end{array} \quad (39)$$

and

$$h = \begin{pmatrix} \Gamma_1 \bar{\phi}_1(\tilde{x}_{1,k},t)(\tilde{x}_{1,k} - z_{2,k}\frac{\partial \psi_1(\tilde{x}_{1,k},t)}{\partial \tilde{x}_{1,k}}) \\ \Gamma_2 z_{2,k} \bar{\phi}_2(\tilde{x}_{1,k}, x_{2,k}, t) \end{pmatrix}, \quad (40)$$

where the subscript k denotes the iteration number. The initial conditions at each iteration must satisfy either the resetting condition, i.e., $\tilde{x}_{1,k}(0) = z_{2,k}(0) = 0$ or the alignment condition i.e., $\tilde{x}_{1,k}(0) = \tilde{x}_{1,k-1}(T)$, $z_{2,k}(0) = z_{2,k-1}(T)$. For instance, the resetting condition is satisfied in the case where $\tilde{x}_{1,k}(0) = 0$, $x_{2,k}(0) = \dot{y}_d(0)$ and $f_1(0) = \phi_1(0) = 0$.
The boundedness and the convergence of $\tilde{x}_{1,k}(t)$ and $z_{2,k}(t)$ are guaranteed as per Theorem 1. Note that θ_1 and θ_2 are allowed to be time-varying in the case $\gamma = 1$. In this example, a second order system was considered, for simplicity. It is worth noting that our approach can be used for a more general class of nonlinear systems stabilizable via adaptive backstepping [9]. We have just to be careful with the initial conditions since new variables are introduced at each step of the backstepping procedure. In fact, the resetting or the alignment condition has to be satisfied for all the variables, other than the parameter estimation errors, involved in the final Lyapunov function.

IV. Conclusion

We proposed a systematic procedure for the design of adaptive ILC schemes for uncertain nonlinear systems based on the existence of a Lyapunov function for the system under consideration. In fact, if a Lyapunov-based standard adaptive control law can be designed and the Lyapunov function satisfies properties $P1$ and $P2$, we show that the extension of the standard adaptive controller to an adaptive ILC controller is straightforward. The resulting parametric adaptation law is quite general in the sense that it includes the pure time-domain adaptation for $\gamma = 0$, the pure iteration-domain adaptation for $\gamma = 1$, and the combination of both for $\gamma \in (0, 1)$. It has been shown that the main advantages of the pure time-domain adaptation is the low memory-size requirement in real-time implementations as well as the simplicity of the design since both properties $P1$ and $P2$ are not required. The pure iteration-domain adaptation is a discrete-type integration along the iteration axis and hence it does not require an approximative numerical integration at each iteration in real time applications, and it does not require the unknown parameters to be time-invariant (as in the case of the pure time-domain or in the case of the combination of both adaptation types). With the pure iteration-domain adaptation, i.e., $\gamma = 1$, we guarantee the boundedness of the infinity norm of the tracking error, the boundedness of the $\mathcal{L}_2$-norm of the control input as well as the convergence to zero of the $\mathcal{L}_2$-norm of the tracking error (under the alignment condition) and the convergence to zero of the infinity norm of the tracking error (under the resetting condition). With the pure time-domain or with the combination of both adaptation types, i.e., $\gamma \in [0, 1)$, we guarantee the boundedness of the infinity norm of all signals as well as the convergence to zero of the infinity norm of the tracking error.

References

[1] S. Arimoto, S. Kawamura and F. Miyazaki, "Bettering operation of robots by learning," Journal Robot. Syst., 1, pp. 123-140, 1984.
[2] S. Arimoto, Control theory of non-linear mechanical systems, Oxford Science Publications, Oxford, UK, 1996.
[3] Z. Bien and J-X. Xu, Iterative learning control: Analysis, Design, Integration and Applications, Kluwer Academic Publishers, 1998.
[4] Y. Chen and C. Wen, Iterative learning control: Convergence, Robustness and Applications, Lecture notes in control and information sciences, Springer Verlag, 1999.

[5] C-J. Chien and C-Y Yao, "Iterative learning of model reference adaptive controller for uncertain nonlinear systems with only output measurement," Automatica, Vol. 40, No. 5, pp. 855-864, 2004.
[6] M. French and E. Rogers, "Nonlinear iterative learning by an adaptive Lyapunov technique," International Journal of Control, Vol. 73, no. 10, pp. 840-850, 2000.
[7] C. Ham, Z.H. Qu and J.H. Kaloust, "Nonlinear learning control for a class of nonlinear systems based on Lyapunov's direct method,". In Proc. American Control Conference, Seattle, USA, pp. 3024-3028, 1995.
[8] C. Ham, Z. Qu and R. Johnson, "Nonlinear learning control for a class of nonlinear systems," Automatica, Vol. 37, pp. 419-428, 2001.
[9] M. Krstic, I. Kanellakopoulos and P. Kokotovic, Nonlinear and adaptive control design, Wiley-Interscience Publication, New York, 1995.
[10] Z. Qu and J. Xu, "Asymptotic learning control for a class of cascaded nonlinear uncertain systems," IEEE Trans. Automat. Contr., vol. 47, no. 8, pp. 13691376, 2002.
[11] K.L. Moore, "Iterative Learning Control: An Expository Overview," Applied and Computational Controls, Signal Processing, and Circuits, Vol. 1, pp. 151-214, 1999.
[12] A. Tayebi, "Adaptive iterative learning control for robot manipulators," Automatica, Vol.40, no.7, pp. 1195-1203, 2004.
[13] J-X. Xu and Y. Tan, Linear and nonlinear iterative learning control, Lecture notes in control and information sciences, Springer Verlag, 2003.
[14] J-X. Xu, "The frontiers of iterative learning control—Part II," Journal of Systems, Control and Information, Vol. 46, no. 5, pp.233-243, 2002.