Proceedings of the
46th IEEE Conference on Decision and Control
New Orleans, LA, USA, Dec. 12-14, 2007

Attitude estimation and stabilization of a rigid body using low-cost sensors

A. Tayebi, S. McGilvray, A. Roberts and M. Moallem

Abstract— **We consider the attitude and gyro-bias estimation as well as the attitude stabilization problem of a rigid body in space using low-cost sensors. We assume that a biased measurement of the angular velocity of the rigid body is provided by a low-cost gyroscope, and the attitude measurements are obtained using low-cost (low-pass) sensors such as accelerometers and magnetometers. We provide an attitude and gyro-bias estimation algorithm using the biased gyro measurement and the attitude measurements—which represent the true attitude only at low frequencies— provided by the low-cost attitude sensors. Finally, the proposed estimation algorithm is coupled with a quaternion-based attitude stabilization scheme, and simulation results are provided to show the effectiveness of the proposed algorithm.**

I. INTRODUCTION

The attitude stabilization problem of a rigid spacecraft or a rigid body in general is an interesting problem in view of the theoretical and practical challenges involved in it. From a pure theoretical point of view, this problem has been solved by several authors in the literatures under the assumption that the angular velocity and the attitude of the rigid body are well known (see for instance [8], [23], [22]). A few controllers have also been tested experimentally in [17]. The problem becomes more challenging if the angular velocity is supposed to be unknown. In this case also, several solutions have been proposed in the literature (see, for instance, [3], [4], [9], [15], [18], [20]). In the case where low-cost gyroscopes, providing biased angular velocity measurements, are used, several authors in the literature proposed estimation algorithms providing an estimation of the constant gyro-bias assuming that the attitude of the rigid body is well known ([5], [10], [19], [21]). In practical applications using low-cost sensors, we generally use low-cost gyroscopes that provide a biased-measurement[1] of the angular velocity of the rigid body; while the attitude is generally obtained by fusing data from low-pass accelerometers and magnetometers which provide a relatively accurate measurement at low frequencies. In [2], [17] a linear complementary filtering approach, assuming small angles variation, has been used to generate the pitch and roll by fusing the measurements provided by the gyroscopes and the tilt-meters in the frequency domain. A nonlinear complementary filtering approach with gyro-bias estimation has also been discussed in [5], [10]. In [13], it is shown that there exists a non-local high gain observer for the roll and pitch angles estimation using low-pass tiltmeters assuming exact knowledge of the angular velocity and low translational accelerations. In [14], a Kalman filter based state estimation algorithm that fuses data from rate gyros (assumed to provide exact angular velocity measurements) and accelerometers, providing long-term drift free attitude estimates, has been derived. The algorithm is a kind of complementary filtering in the sense that the data fusion is based on a switching architecture consisting of two modes (low and high accelerations). In [1], it is shown that it is possible to solve the attitude regulation problem (in a local sense) of a rigid body using rate gyros (assumed to provide exact angular velocity measurements) and low pass accelerometers. The Kalman filter has been used by several authors in the literature attempting to estimate the attitude of a rigid body in space using low-cost sensors. For instance, in [11] an extended Kalman filter has been used for real-time estimation of the attitude of a rigid body using accelerometers, magnetometers and gyroscopes. Although used successfully in certain applications, extended Kalman filters, applied for nonlinear systems, often exhibit an unpredictable behavior.

In this paper, we attempt to go a step further in providing a reasonable solution to the attitude estimation and stabilization of a rigid body in space using low-cost sensors. In fact, assuming that a biased angular velocity measurement can be obtained via a low-cost gyroscope and a measure of the attitude can be obtained using low-pass sensors, we derive an estimation algorithm providing asymptotic estimates for the gyro-bias and the actual attitude of the body over a wide range of frequencies. The main idea is based on the estimation of a 'virtual' angular velocity of the body generating the attitude of the body (which is the true attitude at low frequencies but not at high frequencies) provided by the low-pass sensors. This 'virtual' angular velocity, namely $\bar{\Omega}$, is used in the adaptation law for the bias estimate. Thereafter, a new complementary filter is used to recover the actual attitude. The proposed estimation algorithm has been coupled with a quaternion based PD controller for the attitude stabilization. Simulation results are provided to illustrate the performance of the proposed estimation and control scheme.

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC)

A. Tayebi and A. Roberts are with the Department of Electrical Engineering, Lakehead University, Thunder Bay, Ontario, P7B 5E1, Canada. atayebi@lakeheadu.ca

S. McGilvray and M. Moallem are with the Department of Electrical and Computer Engineering, The University of Western Ontario, London, Ontario Canada N6A 5B9. {smcgilvr,mmoallem}@uwo.ca

[1]The angular velocity is relatively accurate only at high frequencies, i.e., the constant component of the angular velocity (gyro-bias) cannot be measured

II. DYNAMICAL MODEL

The dynamical model of a spacecraft or a rigid body in space is given by

$$I_f\dot{\Omega} = -\Omega \times I_f\Omega + \tau, \quad (1)$$

$$\dot{R} = RS(\Omega), \quad (2)$$

where Ω denotes the angular velocity of the body expressed in the body-fixed frame $\mathcal{A}$. The orientation of the rigid body is given by the orthogonal rotation matrix $R \in SO(3)$. $I_f \in \mathbb{R}^{3\times3}$ is a symmetric positive definite constant inertia matrix of the body with respect to the frame $\mathcal{A}$ whose origin is at the center of mass. The vector τ is the torque applied the rigid body, considered as the input vector. The matrix $S(\Omega)$ is a skew-symmetric matrix such that $S(\Omega)V = \Omega \times V$ for any vector $V \in \mathbb{R}^3$, where $\times$ denotes the vector cross-product.

III. UNIT QUATERNION

The orientation of a rigid body with respect to the inertial frame can be described by a unit quaternion [12]. A quaternion $Q = (q_0, q)$ is composed of a scalar component $q_0 \in \mathbb{R}$ and a vector $q \in \mathbb{R}^3$. The set of quaternion $\mathbb{Q}$ is a four-dimensional vector space over the reals, which forms a group with the quaternion multiplication denoted by "$\star$". The quaternion multiplication is distributive and associative but not commutative [12]. The multiplication of two quaternion $Q = (q_0, q)$ and $P = (p_0, p)$ is defined as [12], [16]

$$Q \star P = (q_0p_0 - q^Tp\ ,\ q_0p + p_0q + q \times p), \quad (3)$$

and has the quaternion $(1, \mathbf{0})$ as the identity element. Note that, for a given quaternion $Q = (q_0, q)$, we have $Q \star Q^{-1} = Q^{-1} \star Q = (1, \mathbf{0})$, where $Q^{-1} = \frac{(q_0,-q)}{\|Q\|^2}$.
The set of unit quaternion $\mathbb{Q}_u$ is a subset of $\mathbb{Q}$ such that

$$\mathbb{Q}_u = \{Q = (q_0, q) \in \mathbb{R} \times \mathbb{R}^3 \mid q_0^2 + q^Tq = 1\}. \quad (4)$$

Note that in the case where $Q = (q_0, q) \in \mathbb{Q}_u$, the unit quaternion inverse is given by $Q^{-1} = (q_0, -q)$.
A rotation matrix R by an angle γ about the axis described by the unit vector $\hat{k} \in \mathbb{R}^3$, can be described by a unit quaternion $Q = (q_0, q) \in \mathbb{Q}_u$ such that

$$q = \hat{k}\sin(\frac{\gamma}{2}), \quad q_0 = \cos(\frac{\gamma}{2}), \quad (5)$$

Algorithms allowing to obtain R from q and q_0 as well as the extraction of q and q_0 from a rotation matrix R, can be found in [6], [16].

In this paper, instead of using the rotation matrix R to describe the orientation of the rigid body in space, we will use the unit quaternion. The dynamical equation (2) can be replaced by the following dynamical equation in terms of the unit quaternion [6], [16]:

$$\dot{Q} = \frac{1}{2}Q \star Q_\Omega, \quad (6)$$

where $Q = (q_0, q) \in \mathbb{Q}_u$ and $Q_\Omega = (0, \Omega) \in \mathbb{Q}$. In the sequel, we will use Q_* to denote the quaternion $(0, *)$.

We also define the unit quaternion error $\tilde{Q}$, which describes the deviation between two unit quaternion Q and $\hat{Q}$, as follows:

$$\tilde{Q} = \hat{Q}^{-1} \star Q = (\hat{q}_0q_0 + \hat{q}^Tq\ ,\ \hat{q}_0q - q_0\hat{q} - \hat{q} \times q) := (\tilde{q}_0, \tilde{q}). \quad (7)$$

IV. PROBLEM STATEMENT

In this paper, our main objective is the attitude estimation and control using low-cost inertial measurement units (IMU). It is clear that if the angular velocity Ω and the initial orientation of the aircraft are exactly known, it is possible to get the instantaneous attitude of the aircraft by integrating (6). Unfortunately, the angular velocity obtained from the gyroscopes are often biased, which prevents the exact recovery of the attitude from (6). Moreover, the attitude provided by the IMU (*i.e.,* for instance, through the fusion of the measurements obtained from magnetometers and accelerometers) is valid only at low frequencies, which is a serious drawback if one seeks the attitude recovery over a wide range of frequencies. In this paper, we aim to provide a solution to this crucial problem by estimating the gyro-bias using the attitude measurements which are valid only at low frequencies as well as gyroscopic measurements. By recovering the gyro-bias, one can recover the angular velocity, and hence, an estimation of the attitude can be obtained, over a wide range of frequencies, using a new complementary filter and integrating (6) under the assumption that initially, the rigid body is at rest or slowly moving[2]. Furthermore, we will show that the estimated attitude and gyro bias can be used in a quaternion-based PD controller for the attitude stabilization of the rigid body in space.

V. GYRO-BIAS ESTIMATION USING EXACT ATTITUDE MEASUREMENTS Q

As we said before, the main problem, in practice, is that low-cost gyroscopes do not provide exact measurements of Ω. One reasonable assumption is to consider

$$\Omega = \Omega_g - b \quad (8)$$

where Ω_g is a measurement provided by the gyroscopes, and b is an unknown finite constant (or slowly varying) bias. Under this assumption, several authors in the literature provided a way to estimate b assuming that Q is available [10], [19]. In fact, in [19], the following estimator, leading to a global exponential convergence of the bias estimation-error to zero, has been proposed:

$$\dot{\hat{Q}} = \frac{1}{2}\hat{Q} \star Q_\beta, \quad (9)$$

with

$$\beta = R^T(\tilde{Q})(\Omega_g - \hat{b} + ksgn(\tilde{q}_0)\tilde{q}), \quad (10)$$

$$\dot{\hat{b}} = -\frac{1}{2}sgn(\tilde{q}_0)\tilde{q}, \quad (11)$$

[2]This assumption allows to use the initial conditions of (6) from the initial attitude provided by the accelerometers and magnetometers which are valid at low frequencies.

where $k > 0$, $\tilde{Q} = Q \star \hat{Q}^{-1} = (\tilde{q}_0, \tilde{q})$ and $R = (\tilde{q}_0^2 - \tilde{q}^T\tilde{q})I + 2\tilde{q}\tilde{q}^T - 2\tilde{q}_0 S(\tilde{q})$. A quite similar estimator (termed passive complementary filter), has also been presented in [10], using the rotation matrix instead of the quaternion.

In this context (*i.e.,* in the case where Q and Ω_g are known), another alternative solution, quite similar to [10], [19], for the estimation of the gyro bias b, which does not involve the term $R^T(\tilde{Q})$ as well as the scalar part of the error-quaternion, is provided in the following proposition:

Proposition 1: Assume that Ω and $\dot{\Omega}$ are bounded and consider the following estimator

$$\begin{array}{rcl} \dot{\hat{Q}} & = & \frac{1}{2}\hat{Q} \star Q_\beta \\ \beta & = & \Omega_g - \hat{b} + \Gamma_1 \tilde{q} \\ \dot{\hat{b}} & = & -\Gamma_2 \tilde{q} \end{array} \tag{12}$$

where $\hat{Q} = (\hat{q}_0, \hat{q}) \in \mathbb{Q}_u$, $Q_\beta = (0, \beta) \in \mathbb{Q}$, $\Gamma_1 = \Gamma_1^T > 0$, $\Gamma_2 = \Gamma_2^T > 0$, and $\tilde{q}$ is the vector part of the unit quaternion deviation defined in (7). Then, $\tilde{Q}$ and $\hat{b}$ are globally bounded and $\lim_{t\to\infty} \hat{b}(t) = b$, $\lim_{t\to\infty} \tilde{q}(t) = 0$ and $\lim_{t\to\infty} \tilde{q}_0(t) = \pm 1$. Furthermore, for all initial conditions such that $(\tilde{Q}(0), \tilde{b}(0)) \neq ((-1, 0, 0, 0), 0)$, we have $\lim_{t\to\infty} \tilde{q}_0(t) = 1$.

Proof: The time derivative of $\tilde{Q}$, in view of (6) and (12), is given by

$$\begin{array}{rcl} \dot{\tilde{Q}} & := & (\dot{\tilde{q}}_0, \dot{\tilde{q}}) \\ & = & \left(\frac{1}{2}\tilde{q}^T(\beta - \Omega)\ ,\ \frac{1}{2}\tilde{q}_0(\Omega - \beta) + \frac{1}{2}\tilde{q} \times (\Omega + \beta)\right) \end{array} \tag{13}$$

Consider the following Lyapunov function candidate

$$\begin{array}{rcl} V & = & \tilde{q}^T\tilde{q} + (\tilde{q}_0 - 1)^2 + \frac{1}{2}\tilde{b}^T\Gamma_2^{-1}\tilde{b} \\ & = & 2(1 - \tilde{q}_0) + \frac{1}{2}\tilde{b}^T\Gamma_2^{-1}\tilde{b}, \end{array} \tag{14}$$

where $\tilde{b}(t) = \hat{b}(t) - b$. The time derivative of (14), in view of (13) and (8), is given by

$$\begin{array}{rcl} \dot{V} & = & -2\dot{\tilde{q}}_0 + \tilde{b}^T\Gamma_2^{-1}\dot{\tilde{b}} \\ & = & -\tilde{q}^T(\beta - \Omega) + \tilde{b}^T\Gamma_2^{-1}\dot{\hat{b}} \\ & = & -\tilde{q}^T(\beta - \Omega_g + b) + \tilde{b}^T\Gamma_2^{-1}\dot{\hat{b}} \end{array} \tag{15}$$

which in view of (12), becomes

$$\dot{V} \quad = \quad -\tilde{q}^T\Gamma_1\tilde{q}. \tag{16}$$

Now, one can conclude that $\tilde{Q}$ and $\tilde{b}$ are globally bounded. One can also show that $\ddot{V}$ is bounded. Hence, $\lim_{t\to\infty} \tilde{q}(t) = 0$, which implies that $\lim_{t\to\infty} \tilde{q}_0(t) = \pm 1$. Since Ω and $\dot{\Omega}$ are bounded, it is clear that $\ddot{\tilde{Q}}$ is bounded, and hence $\lim_{t\to\infty} \dot{\tilde{Q}}(t) = 0$, which in turns, from (13), implies that $\lim_{t\to\infty} (\Omega(t) - \beta(t)) = 0$. Finally, using the fact that $\lim_{t\to\infty} \tilde{q}(t) = 0$, one can conclude from (12) that $\lim_{t\to\infty} (\beta(t) - \Omega_g(t) + \hat{b}(t)) = 0$. Consequently, $\lim_{t\to\infty} (\Omega(t) - \Omega_g(t) + \hat{b}(t)) = 0$, which implies that $\lim_{t\to\infty} \hat{b}(t) = b$.

It is clear that $\dot{V} = 0$ at the following two equilibrium points ($\tilde{q}_0 = \pm 1, \tilde{b} = 0$) and $\dot{V} < 0$ outside these equilibrium points. If initially, the system is at one of these two equilibrium points, it will remain there for all subsequent times. In the case where the system is not at one of the two equilibrium points, it will converge to the attractive equilibrium point ($\tilde{q}_0 = 1, \tilde{b} = 0$) for which $V = 0$ and $\dot{V} = 0$. The equilibrium point ($\tilde{q}_0 = -1, \tilde{b} = 0$) is not an attractor, but a repeller [8]. In fact, if the system is initially at $\tilde{q}_0 = -1$ and we apply a small disturbance (preserving the condition $-1 \leq \tilde{q}_0 \leq 1$), one can see from (14) that V decreases, and since $\dot{V} < 0$ outside the equilibrium points, one can conclude that $\tilde{q}_0$ will converge to 1. □

VI. Main Results

A. Gyro-bias estimation using low-cost sensors

In this section, we will assume that the angular velocity is given by (8), and gyroscopes are used to provide Ω_g. We will also assume that the orientation of the rigid body is obtained using low-cost sensors, such as accelerometers and magnetometers, which generally provide accurate attitude measurements at low frequencies. In this case, just an estimate of the rotation matrix $\bar{R}$, which is generally a good estimation of the actual rotation matrix R at low frequencies, is available. From the rotation matrix estimate $\bar{R}$, one can get the quaternion estimate $\bar{Q}$ whose dynamics, reflecting the low pass property of the low-cost sensors, is given by

$$\dot{\bar{Q}} = \frac{1}{2}\bar{Q} \star Q_{\bar{\Omega}}, \quad \text{with} \quad \bar{Q}(0) = Q(0). \tag{17}$$

where $Q_{\bar{\Omega}}$ denotes the quaternion $(0, \bar{\Omega})$, and $\bar{\Omega}$ is a "virtual" angular velocity leading to the measurement $\bar{Q}$. We assume that $\bar{\Omega}$ is related to Ω through the following first order low-pass filter

$$\dot{\bar{\Omega}} = -A\bar{\Omega} + A\Omega, \tag{18}$$

where A is a 3×3 diagonal positive definite matrix, namely $A = \text{diag}\{a_1, a_2, a_3\}$, defining the dynamics of the low-cost sensors used to measure $\bar{Q}$. The positive constants a_1, a_2 and a_3 describe the cut-off frequencies of the attitude sensors. It is clear that $\bar{\Omega}$ coincides with Ω at low frequencies, which implies that $\bar{Q}$ coincides with Q at low frequencies.

The dynamical equation (17) can also be written as follows

$$\dot{\bar{Q}} = \frac{1}{2}E(\bar{Q})\bar{\Omega}, \tag{19}$$

$$E(\bar{Q}) = \begin{pmatrix} -\bar{q}_1 & -\bar{q}_2 & -\bar{q}_3 \\ \bar{q}_0 & -\bar{q}_3 & \bar{q}_2 \\ \bar{q}_3 & \bar{q}_0 & -\bar{q}_1 \\ -\bar{q}_2 & \bar{q}_1 & \bar{q}_0 \end{pmatrix} \tag{20}$$

where $\bar{Q} = (\bar{q}_0,\ \bar{q}_1,\ \bar{q}_2,\ \bar{q}_3)$, $E^T(\bar{Q})E(\bar{Q}) = I_{3\times 3}$, with $I_{3\times 3}$ being the 3×3 identity matrix. Hence from (19), one can obtain $\bar{\Omega}$ as follows:

$$\bar{\Omega} = 2E^T(\bar{Q})\dot{\bar{Q}}, \tag{21}$$

Now, our objective is to derive an estimation algorithm for the gyro bias b, using only Ω_g and $\bar{Q}$.

Proposition 2: Consider the following estimator

$$\dot{\hat{\bar{\Omega}}} = -A\hat{\bar{\Omega}} + A(\Omega_g - \hat{b}), \tag{22}$$

$$\dot{\hat{b}} = -\Gamma\tilde{\Omega}. \tag{23}$$

where $\tilde{\Omega} = \bar{\Omega} - \hat{\bar{\Omega}}$ and $\Gamma = \Gamma^T > 0$. Then $\tilde{b}$ is bounded, and $\lim_{t\to\infty} \hat{b}(t) = b$ and $\lim_{t\to\infty} \tilde{\Omega}(t) = 0$. The convergence of the signals is exponential.

Proof: Let us consider the following Lyapunov function candidate:

$$V = \frac{1}{2}\tilde{b}^T\Gamma^{-1}\tilde{b} + \frac{1}{2}\tilde{\Omega}^T A^{-1}\tilde{\Omega} \tag{24}$$

where $\tilde{b}(t) = \hat{b}(t) - b$. Using the fact that

$$\dot{\tilde{\Omega}} = -A\tilde{\Omega} + A\tilde{b}, \tag{25}$$

the time derivative of (24) is given by

$$\dot{V} = \tilde{b}^T\Gamma^{-1}\dot{\hat{b}} - \tilde{\Omega}^T\tilde{\Omega} + \tilde{\Omega}^T\tilde{b}, \tag{26}$$

which in view of (23), becomes

$$\dot{V} = -\tilde{\Omega}^T\tilde{\Omega}. \tag{27}$$

This implies that $\tilde{b}$ and $\tilde{\Omega}$ are bounded. One can also show that $\ddot{V}$ is bounded. Therefore, $\lim_{t\to\infty} \tilde{\Omega}(t) = 0$. Since $\ddot{\tilde{\Omega}}$ is bounded, it is clear that $\lim_{t\to\infty} \dot{\tilde{\Omega}}(t) = 0$, which, from (25), implies that $\lim_{t\to\infty} \tilde{b}(t) = 0$. Finally, the exponential convergence property can be easily deduced from the fact that the estimation error dynamics is given by

$$\begin{pmatrix} \dot{\tilde{\Omega}} \\ \dot{\tilde{b}} \end{pmatrix} = \begin{pmatrix} -A & A \\ -\Gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{\Omega} \\ \tilde{b} \end{pmatrix} \tag{28}$$

□

Now, let us assume that the positive definite diagonal matrix A is unknown. In this case, we propose the following algorithm which, roughly speaking, relies on the richness of the measured angular velocity signal

Proposition 3: Consider the following estimator

$$\dot{\hat{\bar{\Omega}}} = -\bar{A}\hat{\bar{\Omega}} + \bar{A}(\Omega_g - \hat{b}) + M(t)\hat{\theta}, \tag{29}$$

$$\dot{\hat{b}} = -\Gamma\tilde{\Omega}. \tag{30}$$

$$\dot{\hat{\theta}} = \bar{\Gamma}M(t)\tilde{\Omega}. \tag{31}$$

where $\tilde{\Omega} = \bar{\Omega} - \hat{\bar{\Omega}}$, $\bar{\Gamma} = \bar{\Gamma}^T > 0$, Γ is a diagonal positive definite matrix and $\bar{A}$ is a diagonal positive semi-definite matrix. The matrix $M(t)$ is given by $M(t) = \text{diag}\{m_1(t), m_2(t), m_3(t)\}$, where m_1, m_2 and m_3 are the three components of the vector $(\Omega_g - \hat{\bar{\Omega}} - \hat{b})$. Assume that Ω_g and b are bounded. Then $\tilde{\Omega}$, $\hat{b}$ and $\hat{\theta}$ are bounded and $\lim_{t\to\infty} \tilde{\Omega}(t) = 0$. Moreover, if $F(t) \triangleq [A, -M(t)]^T$ satisfies the following persistency of excitation condition

$$\sigma_1 I_{6\times6} \leq \frac{1}{T}\int_t^{t+T} F(\tau)F^T(\tau)d\tau \leq \sigma_2 I_{6\times6}, \quad \forall t \tag{32}$$

for some positive constants σ_1, σ_2 and T, then $\tilde{\Omega}$, $(\hat{b} - b)$ and $(\hat{\theta} - \theta)$ converge exponentially to zero.

Proof: Let us consider the following Lyapunov function candidate:

$$V = \frac{1}{2}\tilde{b}^T A\Gamma^{-1}\tilde{b} + \frac{1}{2}\tilde{\theta}^T\bar{\Gamma}^{-1}\tilde{\theta} + \frac{1}{2}\tilde{\Omega}^T\tilde{\Omega} \tag{33}$$

where $\tilde{b}(t) = \hat{b}(t) - b$ and $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$, with $\theta = [\theta_1, \theta_2, \theta_3]^T$ being a vector whose elements are the diagonal elements of $\Delta A = A - \bar{A} = \text{diag}\{\theta_1, \theta_2, \theta_3\}$. Using the fact that

$$\begin{aligned} \dot{\tilde{\Omega}} &= -A\tilde{\Omega} + A\tilde{b} + \Delta A(\Omega_g - \hat{\bar{\Omega}} - \hat{b}) - M(t)\hat{\theta} \\ &= -A\tilde{\Omega} + A\tilde{b} - M(t)\tilde{\theta}, \end{aligned} \tag{34}$$

the time derivative of (33) is given by

$$\dot{V} = \tilde{b}^T A\Gamma^{-1}\dot{\hat{b}} + \tilde{\theta}^T\bar{\Gamma}^{-1}\dot{\hat{\theta}} + \tilde{\Omega}^T(-A\tilde{\Omega} + A\tilde{b} - M(t)\tilde{\theta}) \tag{35}$$

which in view of (30) and (31), becomes

$$\dot{V} = -\tilde{\Omega}^T A\tilde{\Omega} = -X^T C^T C X, \tag{36}$$

where $X = [\tilde{\Omega}^T\ \tilde{b}^T\ \tilde{\theta}^T]^T$ and $C = [A^{1/2}\ 0_{3\times3}\ 0_{3\times3}]$ with $A^{1/2}A^{1/2} = A$. This implies that $\tilde{b}$, $\tilde{\theta}$ and $\tilde{\Omega}$ are bounded. The error dynamics is given by

$$\dot{X} = N(t)X, \tag{37}$$

with

$$N(t) = \begin{pmatrix} -A & A & -M(t) \\ -\Gamma & 0 & 0 \\ \bar{\Gamma}M(t) & 0 & 0 \end{pmatrix} \tag{38}$$

Hence, system (37) is exponentially stable if the pair $(N(t), C)$ is uniformly completely observable (UCO) [7]. The pair $(N(t), C)$ is UCO if the pair $(N(t) - K(t)C, C)$ is UCO for some $K(t) \in \mathcal{L}_\infty$. Picking $K(t) = [\gamma A^{-1/2} - A^{1/2},\ -\Gamma,\ -\bar{\Gamma}M(t)]^T$ for some $\gamma > 0$, we obtain

$$\begin{aligned} N(t) - K(t)C &= \begin{pmatrix} -\gamma I & A & -M(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma I & F(t) \\ 0_{6\times3} & 0_{6\times6} \end{pmatrix} \end{aligned} \tag{39}$$

Finally, condition (32) follows from ([7], Lemma 4.8.4, Lemma 5.6.3). □

Remark 1: Note that $\bar{\Omega}$, required in the algorithm proposed in Proposition 2 and Proposition 3, cannot be obtained from (18) since Ω is unknown. On the other hand, since the derivative of $\bar{Q}$ is not available, a practical solution is to substitute $\dot{\bar{Q}}$ in (21) by the so-called 'dirty derivative'

$$[\dot{\bar{Q}}(t)] \approx \frac{s}{1 + \bar{\tau}s}[\bar{Q}(t)] \tag{40}$$

where $\frac{1}{\bar{\tau}}$ is the cut-off frequency of the low-pass filter.

B. Complementary filtering

It is important to notice that, although the gyro-bias could be recovered with just the use of Ω_g and $\bar{Q}$, it is not guaranteed that the real attitude Q could be recovered by integrating

$$\begin{aligned} \dot{\hat{Q}} &= \frac{1}{2}\hat{Q} \star Q_\beta, \quad \hat{Q}(0) = \bar{Q}(0), \\ \beta &= \Omega_g - \hat{b}. \end{aligned} \tag{41}$$

In fact, even though $\hat{b}(t)$ converges exponentially to b (*i.e.,* $\beta(t)$ converges exponentially to $\Omega(t)$), and $\hat{Q}(0) = Q(0)$, one cannot precisely recover the real attitude Q when t goes to infinity. Nevertheless, one can show that a fast convergence of $\tilde{b}$ and a small magnitude of the angular velocity, lead to a small error between the estimated quaternion $\hat{Q}$ and the real quaternion Q. In fact, one has

$$\begin{array}{l}\|Q(t) - \hat{Q}(t)\| \leq \|Q(0) - \hat{Q}(0)\| \\ +\frac{1}{2}\int_0^t \|Q(\tau)\star Q_\Omega(\tau) - \hat{Q}(\tau)\star Q_\beta(\tau)\|d\tau \\ \leq \|Q(0) - \hat{Q}(0)\| + \frac{1}{2}\int_0^t \|Q(\tau) - \hat{Q}(\tau)\|\|\Omega(\tau)\|d\tau \\ +\frac{1}{2}\int_0^t \|\tilde{b}(\tau)\|d\tau \end{array} \tag{42}$$

where the property $\|Q\star Q_\Omega\| \leq \|\Omega\|$ has been used. Since $\hat{Q}(0) = Q(0)$ and $\tilde{b}$ converges exponentially to zero, inequality (42) leads to

$$\lim_{t\to\infty} \|Q(t) - \hat{Q}(t)\| \leq \min\left\{2\ ,\ \int_0^\infty \|\Omega(\tau)\|d\tau + \epsilon\right\}, \tag{43}$$

where $\epsilon = \frac{1}{2}\int_0^\infty \|\tilde{b}(\tau)\|d\tau$. Applying Gronwall-Bellman lemma, one can also derive the following inequality from (42)

$$\lim_{t\to\infty} \|Q(t) - \hat{Q}(t)\| \leq \min\left\{2\ ,\ \epsilon \exp\left[\frac{1}{2}\int_0^\infty \|\Omega(\tau)\|d\tau\right]\right\}, \tag{44}$$

On the other hand, since the measurement $\bar{Q}$ is reliable at low frequencies, it is natural to rely on that instead of $\hat{Q}$ from (41) at low frequencies. Therefore, we propose the following complementary filter based quaternion estimation:

$$\begin{array}{rcl}\dot{\hat{Q}} &=& \frac{1}{2}\hat{Q}\star Q_\beta, \qquad \hat{Q}(0) = \bar{Q}(0), \\ \beta &=& F_1(s)[\Omega_g - \hat{b}] + F_2(s)[\bar{\Omega} + \bar{\Gamma}\tilde{q}]\end{array} \tag{45}$$

where $\bar{\Gamma} = \bar{\Gamma}^T > 0$, $\tilde{q}$ is the vector part of $\tilde{Q} = \hat{Q}^{-1}\star\bar{Q}$, and $\hat{b}(t)$ is bounded and converges exponentially to b (such as in proposition 2 and proposition 3). $F_1(s)$ and $F_2(s)$ are two complementary filters such that $F_1(s) + F_2(s) = 1$. At low frequencies, $F_2(s)$ is close to one allowing to select $\beta = \bar{\Omega} + \bar{\Gamma}\tilde{q}$ which forces the solution $\hat{Q}$ to converge to $\bar{Q}$ from any initial condition as shown below. Taking the following Lyapunov function candidate:

$$\begin{array}{rcl} V &=& \tilde{q}^T\tilde{q} + (\tilde{q}_0 - 1)^2 \\ &=& 2(1 - \tilde{q}_0)\end{array} \tag{46}$$

and using the fact that

$$\begin{array}{rcl}\dot{\tilde{Q}} &=& \frac{d}{dt}(\hat{Q}^{-1}\star\bar{Q}) \\ &=& \left(\frac{1}{2}\tilde{q}^T(\beta - \bar{\Omega})\ ,\ \frac{1}{2}\tilde{q}_0(\bar{\Omega} - \beta) + \frac{1}{2}\tilde{q}\times(\bar{\Omega} + \beta)\right) \\ &:=& (\dot{\tilde{q}}_0, \dot{\tilde{q}}),\end{array} \tag{47}$$

we obtain, in view of (47) and the fact $\beta = \bar{\Omega} + \bar{\Gamma}\tilde{q}$, the following result

$$\dot{V} = -\tilde{q}^T\bar{\Gamma}\tilde{q}. \tag{48}$$

VII. Attitude stabilization using low-cost sensors

It is well known that the following control law

$$\tau = -\alpha_1 q - \alpha_2\Omega, \tag{49}$$

where α_1 is a positive scalar and α_2 is a symmetric positive definite matrix, guarantees global asymptotic stability of the equilibrium point $(R = I, \Omega = 0)$ as shown, for instance, in [8], [9], [22], [23]. The equilibrium point $(R = I, \Omega = 0)$ for (1) and (2) is equivalent to the equilibrium point $(q = 0, q_0 = \pm 1, \Omega = 0)$ for (1) and (6). Note that the two equilibrium points $(q = 0, q_0 = \pm 1, \Omega = 0)$ are in reality a unique physical equilibrium point corresponding to $(R = I, \Omega = 0)$. The vector quaternion q and the angular velocity Ω could be replaced, respectively, by $\hat{q}$ given by (45) and $(\Omega_g - \hat{b})$ given in Proposition 2 or Proposition 3.

VIII. Simulation results

A. Simulation results for Proposition 2

The inertia matrix is taken as $I_f = \mathrm{diag}\{4.9e-3, 4.9e-3, 8.8e-3\}Kg.m^2$, and the unknown gyro bias is taken as $b = [1, 0.5, -0.5]^T rad/s$. We applied the control law $\tau = -\alpha_1\hat{q} - \alpha_2(\Omega_g - \hat{b})$, where by $\hat{q}$ is given by (45) and $\hat{b}$ as given by Proposition 2. The initial conditions have been taken as follows: $Q(0) = \hat{Q}(0) = \bar{Q}(0) = (0.3919, -0.2006, 0.532, 0.7233)$ corresponding to initial Euler angles of $(\pi/6, -\pi/4, \pi/3)$, $\Omega(0) = \hat{b}(0) = (0\ \ 0\ \ 0)^T$. The gain Γ has been taken as $\Gamma = 30I_{3\times 3}$. The control gains have been taken as $\alpha_1 = 0.5$ and $\alpha_2 = 0.5\ I_{3\times 3}$. The cut-off frequency of the filtered derivative, used to get $\bar{\Omega}$, is taken as $\frac{1}{\tau} = 10^3 rad/s$. We assume that the quaternion $\bar{Q}$ is obtained using low-cost sensors such as $\bar{Q}$ is an accurate description of Q for frequencies less than $\frac{3}{2\pi}Hz$. This is taken into consideration through the choice of the matrix $A = 3I_{3\times 3}$. The gain in the complementary filter has been chosen as $\bar{\Gamma} = 20I_{3\times 3}$ and the transfer functions used in the complementary filter have been taken as:

$$F_1(s) = \frac{s^2}{s^2 + 2\xi\omega_n s + w_n^2}, \quad F_2(s) = \frac{2\xi\omega_n s + w_n^2}{s^2 + 2\xi\omega_n s + w_n^2}$$

with $\xi = 0.7$ and $w_n = 3$.

Figure 1 shows the evolution of the adapted parameters function of time. Figure 2 shows the evolution of the actual quaternion $q = (q_0, q_1, q_2, q_3)$ and its estimation $\hat{q}$ function of time. Figure 3 shows the evolution of the angular velocity Ω function of time.

IX. Conclusion

An attitude and gyro-bias estimation algorithm, based on the measurements provided by low-cost sensors, has been proposed for a rigid body in space. We assume that the real angular velocity is equal to the velocity provided by the gyroscope plus a certain unknown constant bias. We also assume that the attitude sensors provide measurements that are true only at low frequencies. Using a new quaternion based complementary filter, our algorithm is able to asymptotically recover the actual attitude of the rigid body over a wide

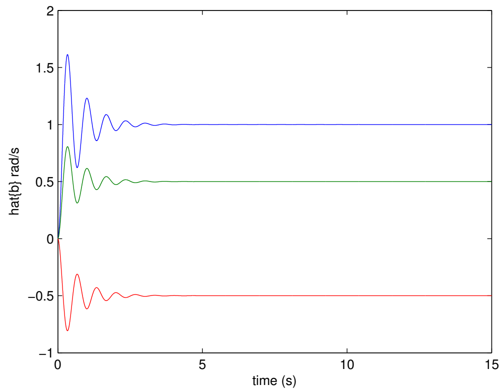

Fig. 1. The adapted parameter $\hat{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ versus time, for Proposition 2

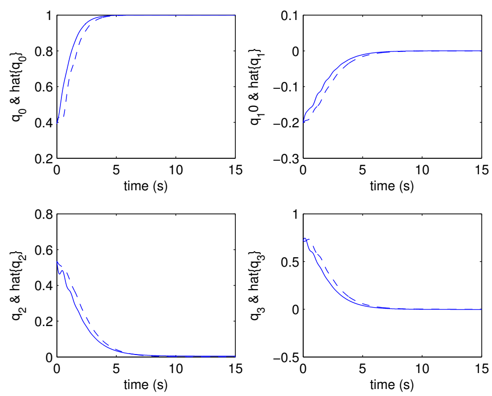

Fig. 2. Actual Q (solid) and its estimation $\hat{Q}$ (dashed) versus time, for Proposition 2

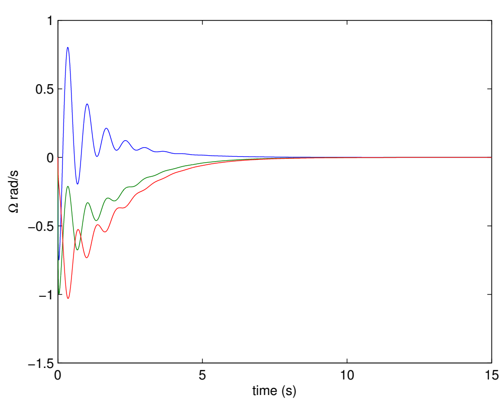

Fig. 3. The angular velocity Ω versus time, for Proposition 2

range of frequencies. The proposed estimation algorithm has been coupled with a quaternion-based PD controller and simulation results have been provided. For space limitation reason, we presented simulation results just for proposition 2.

References

[1] M.R. Akella, J.T. Halbert and G.R. Kotamraju, "Rigid body attitude control with inclinometer and low-cost gyro measurements," *Systems & Control Letters*, Vol. 49, pp. 151-159, 2003.

[2] A.J. Baerveldt and R. Klang, "A low-cost and low-weight attitude estimation system for an autonomous helicopter," *Proc. IEEE Int. Conf. on Intelligent Engineering Systems*, pp. 391-395, 1997.

[3] F. Caccavale and L. Villani, "Output feedback control for attitude tracking ," *Systems & Control Letters*, Vol. 38, No. 2, pp. 91-98, 1999.

[4] O. Egeland and J.M. Godhavn, "Passivity-based adaptive attitude control of a rigid spacecraft," *IEEE Transactions on Automatic Control*, Vol. 39, pp. 842-846, 1994.

[5] T. Hamel and R. Mahony, "Attitude estimation on SO(3) based on direct inertial measurements," To appear in *International Conference on Robotics and Automation*, 2006.

[6] P.C. Hughes, *Spacecraft attitude dynamics*. New York: Wiley, 1986.

[7] P.A. Ioannou and J. Sun, *Robust adaptive control*, Prentice Hall, 1996.

[8] Joshi S. M., A.G. Kelkar and J. T-Y Wen, "Robust Attitude stabilization of spacecraft using nonlinear quaternion feedback," *IEEE Transactions on Automatic Control*, Vol. 40, No. 10, pp. 1800-1803, 1995.

[9] F. Lizarralde and J. T. Wen, "Attitude control without angular velocity measurement: A passivity approach," *IEEE Transactions on Automatic Control*, Vol. 41, No. 3, pp. 468-472, 1996.

[10] R. Mahony, T. Hamel and J-M. Pflimlin, "Complementary filter design on the special orthogonal group $SO(3)$," In proc. of the *44th IEEE CDC, and ECC*, Seville, Spain, pp. 1477-1484, 2005.

[11] J.L. Marins, X. Yun, E.R. Backmann, R.B. McGhee, and M.J. Zyda, "An extended Kalman filter for quaternion-based orientation estimation using marg sensors," In proc. of the *44th IEEE/RSJ IROS*, pp. 2003-2011, 2001.

[12] R.M. Murray, Z. Li and S. Satry.*A mathematical introduction to robotic manipulation*. CRC Press, 1994.

[13] H. Rehbinder and X. Hu, "Nonlinear state estimation for rigid body motion with low-pass sensors," *Systems and Control Letters*, Vol. 40, No. 3, pp. 183-190, 2000.

[14] H. Rehbinder and X. Hu, "Drift-free attitude estimation for accelerated rigid bodies," *Automatica*, Vol. 40, No. 4, pp. 653-659, 2004.

[15] S. Salcudean, "A globally convergent angular velocity observer for rigid body motion," *IEEE Transactions on Automatic Control*, Vol. 36, No. 12, pp. 1493-1497, 1991.

[16] M.D. Shuster, "A survey of attitude representations," *J. Astronautical Sciences*, Vol. 41 , No. 4, pp. 439-517, 1993.

[17] A. Tayebi and S. McGilvray, "Attitude stabilization of a quadrotor aircraft," *IEEE Transactions on Control Systems Technology*, Vol. 14, No. 3, pp. 562-571, 2006.

[18] A. Tayebi, "Unit quaternion observer based attitude stabilization of a rigid spacecraft without velocity measurement," In proc. of the *45th IEEE Conference on Decision and Control*, San Diego, California, USA, 2006, pp. 1557-1561.

[19] J. Thienel and R.M. Sanner "A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise," *IEEE Transactions on Automatic Control*, Vol. 48, No. 11, pp. 2011-2015, 2003.

[20] P. Tsiotras, "Further results on the attitude control problem," *IEEE Transactions on Automatic Control*, Vol. 43, No. 11, pp. 1597-1600, 1998.

[21] B. Vik, T.A. Fossen, "Nonlinear observer design for integration of DGPS and INS," *New Directions in Nonlinear Observer Design. Lecture notes in Control and Information Sciences*, Springer-Verlag, New York, pp. 135-159, 1999.

[22] J. T-Y. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Transactions on Automatic Control*, Vol. 36, No. 10, pp. 1148-1162, 1991.

[23] B. Wie, H. Weiss and A. Arapostathis, "Quaternion feedback regulator for spacecraft eigenaxis rotations," *AIAA J. Guidance Control*, Vol. 12, No. 3, pp. 375-380, 1989.