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Position Control of VTOL UAVs using IMU and GPS measurements

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Abstract—**We propose a new position control scheme for vertical take-off and landing (VTOL) unmanned airborne vehicles (UAVs). Traditionally, the control schemes for this type of vehicle assume that the system attitude is accurately known (or measured). Unfortunately, there does not exist any sensor that directly measures the orientation of a rigid body. Instead, to obtain the orientation of the aircraft, separately designed attitude-estimation schemes relying, for instance, on an inertial measurement unit (IMU) must be employed. Consequently, one drawback of this common practice, on top of the possible inaccuracies in recovering the systems attitude, is mainly related to the difficulty of proving the stability of the overall closed-loop system (observer-controller). Motivated by this problem, we propose a new position control scheme that does not require the recovery of the system's attitude. Instead, we rely on a direct use of the vector measurements provided by the IMU. Our approach can efficiently handle large linear accelerations, which is not the case in traditional controllers relying on IMU-based attitude observers that assume that the accelerometer provides a measurement of the gravity vector in the body-attached frame.**

I. INTRODUCTION

Vertical take-off and landing (VTOL) unmanned airborne vehicles (UAVs) are more commonly being sought to perform a number of tasks, including surveillance, structure inspection, and a variety of other applications where human presence is either difficult or dangerous to achieve. One of the limitations associated with the operation of these aircraft is attributed to the requirement of skilled and trained operators to pilot the system. This limits the usefulness of these systems to persons without the required training. Fortunately, this problem has motivated several groups in the research community to develop flight-control systems for these types of aircraft which allow the aircraft to operate more autonomously, thereby reducing the required skill of the pilot or operator. By relieving the operator of some of the pilot-related duties, more attention can be placed on viewing a camera or other sensors to address the primary mission objective, instead of primarily concentrating on flying the aircraft. As a result of these efforts the research community has seen substantial and interesting advancements in the design of position controllers, for example see [1], [2], [3], [4], [5], [6], [7], [8].

The existing controllers are designed to provide the necessary control input (in terms of the system angular velocity or control-torque that is applied to the rotational dynamics of the system) based upon measurements obtained using a common-sensor set. These sensors usually include a global positioning system (GPS) (to obtain the system position and velocity), and a sensor to provide the attitude of the aircraft (with respect to a fixed inertial frame). However, there does not exist any inertial sensor (to our knowledge) that directly measures the orientation of a rigid body with respect to an inertial frame. In fact, the commercially available *orientation sensors* employ Kalman filtering or some other attitude observer in order to provide the orientation of the aircraft. Therefore, there can be un-modeled dynamics or other errors associated with the filter (or attitude observer) that may not be accounted for by the proposed control scheme.

Due to the unavailability of such a sensor which directly measures orientation, the research community has also been motivated to develop a number of attitude-observers. Typically, the existing attitude-observers utilize a set of vectors (usually assumed to be known in the inertial frame) which are measured in the body-fixed frame. Some examples of attitude observers which use vector measurements can be found in [9], [10], [11], [12], [13] and [14]. However, the main limitation of these types of observers is due to the limited number of sensors that can measure the body-referenced coordinates of a vector which is known in the inertial frame. Typically, the two sensors used most commonly in this capacity are the accelerometer and magnetometer, which due to developments in technologies such as Integrated Micro-Electrical-Mechanical systems (IMEMs), are small, inexpensive, and widely available. The magnetometer is used to measure the known inertially-referenced ambient magnetic field in the body-fixed frame, where the accelerometer is typically used to measure the body-referenced gravity vector. Herein lies a problem since the accelerometer also measures forces due to the acceleration of the device which are likely significant due to aircraft motion, especially in the context of position control.

To address the problem associated with the accelerometers dependance on linear accelerations, a different type of attitude observer has been previously proposed which uses a GPS in addition to the vector measurements from the accelerometer and magnetometer. Examples of this type of observers can be found in [15], [16], and [17]). These so-called *velocity-aided attitude observers*, alternatively use the accelerometer to measure the *apparent acceleration* of the system, rather than assuming only gravity is measured, resulting in significant improvements in performance when the system experiences relatively large linear accelerations.

A second challenge associated with the position control

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problem is due to the fact that practitioners have no other option but to couple an attitude observer with one of the proposed control schemes. Due to the fact that the existing controllers assume that the orientation is directly measured (accurately known), rather than obtained from an attitude observer, there are currently no guarantees for stability (to our knowledge) when these two systems are coupled together. To address this problem, a velocity-free attitude stabilization control scheme that uses direct vector measurements (without attitude-estimation) has been proposed in [13]. A position tracking controller has also been proposed in [18], where the control laws were derived using the vector measurements as inputs (rather than the system orientation), thereby eliminating the need for an attitude observer. However, since this work assumed that the inertial vectors are accurately known in the inertial frame, the use of an accelerometer to measure the gravity vector in the body frame may lead to unexpected performance, especially when the controller demands significant accelerations of the system, thereby affecting the accelerometer signals.

In this paper, we present a new type of position control, which avoids the use of an attitude observer by using the vector measurements (accelerometer and magnetometer) directly in the control scheme. However, motivated by the *velocity aided attitude observers*, in this work we also use the accelerometer to measure the system translational acceleration, instead of assuming only the gravity vector is measured, which (to our knowledge) has not been previously achieved in the available literature in this context. As a result of our proposed control laws, we show that for an appropriate choice of control gains, the system position is guaranteed to converge to a target position which is constant (or slowly varying with respect to time), for almost all initial conditions. This resulting control scheme is likely to provide better performance since it does not assume that the orientation is accurately known, does not require the use of an attitude observer, and is not negatively affected by large system accelerations which could otherwise destroy the performance of previous observers/controllers that use the accelerometer to measure the gravity vector.

II. Background

A. Attitude Representation

Let $\mathcal{I}$ denote an inertial frame of reference rigidly attached to the earth (assumed flat), and $\mathcal{B}$ denote a frame of reference rigidly attached to the aircraft center of gravity. To describe the rotation from $\mathcal{I} \to \mathcal{B}$ we use the quaternion $Q = (\eta, q)$, $\eta \in \mathbb{R}$, $q \in \mathbb{R}^3$, where Q belongs to the set of unit quaternion

$$Q = (\eta, q) \in \mathbb{Q} := \{Q \in \mathcal{S}^3 \mid \|Q\| = 1\}, \tag{1}$$

where $\mathcal{S}$ denotes a three-dimensional sphere ([19], [20],[21]). The unit norm constraint of the quaternion implies $\eta^2 + q^\mathsf{T} q = 1$. The rotation $\mathcal{I} \to \mathcal{B}$ can also be described using a direct cosine (rotation) matrix $R(\eta, q) \in SO(3)$ where $SO(3)$ is the special-orthogonal group

$$SO(3) := \{R \in \mathbb{R}^{3\times 3} \mid \det R = 1 \mid RR^T = R^T R = I_{3\times 3}\}. \tag{2}$$

The rotation matrix $R(\eta, q)$ corresponding to the unit quaternion $Q = (\eta, q)$ can be determined using the so-called Rodrigues formula

$$R(\eta, q) \quad = \quad I_{3\times 3} + 2S(q)^2 - 2\eta S(q), \tag{3}$$

where $S(\cdot)$ is the *skew-symmetric matrix*

$$S(u) = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}, \tag{4}$$

where $u = [u_1, u_2, u_3]^\mathsf{T}$. The set $\mathbb{Q}$ requires the use of some unique mathematical operations which we will define now. Given $Q, P \in \mathbb{Q}$ where $P = (p_0, p)$ the quaternion product is defined by

$$Q \odot P \quad = \quad \big(p_0\eta - q^T p, \quad \eta p + p_0 q + S(q)p\big). \tag{5}$$

Note that the quaternion product of two unit-quaternion preserves the properties of the unit quaternion, i.e $Q \odot P \in \mathbb{Q}$. The unit-quaternion inverse is given by $Q^{-1} = (\eta, -q)$ which has the property $Q \odot Q^{-1} = Q^{-1} \odot Q = (1, \mathbf{0})$ and $Q \odot (1, \mathbf{0}) = (1, \mathbf{0}) \odot Q = Q$, where $(1, \mathbf{0})$ is known as the *identity quaternion*.

B. Attitude Dynamics

Let ω denote the *body-referenced angular velocity* of the frame $\mathcal{B}$ wrt $\mathcal{I}$ (expressed in $\mathcal{B}$). Using the body-referenced angular velocity an expression for the time-derivative of the quaternion Q is given by

$$\dot{Q} = \frac{1}{2} Q \odot \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -q^\mathsf{T} \\ \eta I_{3\times 3} + S(q) \end{bmatrix} \omega. \tag{6}$$

Similarly, an expression for the time-derivative of the rotation matrix $R(\eta, q)$ is given by $\dot{R}(\eta, q) = -S(\omega) R(\eta, q)$.

C. Bounded Functions

Consider a bounded, differentiable function, denoted as $h(\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$, where we also denote $\phi(u) := \frac{\partial}{\partial u} h(u)$, which satisfies the following properties:

$$\begin{array}{ll} u^\mathsf{T} h(u) > 0 & \forall u \in \mathbb{R}^3, \|u\| \in (0, \infty) \\ \left.\begin{array}{c} 0 \le \|h(u)\| < 1 \\ 0 < \|\phi(u)\| \le 1 \end{array}\right\} & \forall u \in \mathbb{R}^3, \|u\| \in [0, \infty). \end{array} \tag{7}$$

Throughout the paper we make use of one example of this type of function which is given by $h(u) = \big(1 + u^\mathsf{T} u\big)^{-1/2} u$ from which one can derive the expression for $\phi(u)$ to be

$$\phi(u) \quad = \quad \big(1 + u^\mathsf{T} u\big)^{-3/2} \big(I_{3\times 3} - S(u)^2\big). \tag{8}$$

D. System Model

Let $p, v \in \mathbb{R}^3$ denote the position and velocity, respectively, of the vehicle COG expressed in the inertial frame $\mathcal{I}$. Let the unit quaternion $Q = (\eta, q)$ represent the rotation between the frame $\mathcal{I}$ and $\mathcal{B}$ which is rigidly attached to the vehicle COG. Let $R = R(\eta, q)$ denote the rotation matrix corresponding to the quaternion Q. Using the framework outlined in section

II-B we consider the following well-known model for a VTOL UAV ([5], [7], [8])

$$\dot{p} = v, \tag{9}$$

$$\dot{v} = \mu + \delta(v), \quad \mu = ge_3 - u_t R^{\mathsf{T}} e_3, \tag{10}$$

$$\dot{Q} = \frac{1}{2}\begin{bmatrix} -q^{\mathsf{T}} \\ \eta I_{3\times 3} + S(q) \end{bmatrix} \omega, \tag{11}$$

where $u_t = T/m_b$, T is the system thrust, m_b is the system mass, $e_3 = \mathrm{col}\,[0, 0, 1]$, g is the gravitational acceleration, and $\delta(v)$ is a disturbance which is dependant on aerodynamic drag forces. The control input of the system is defined as $u = [u_t, \omega]^{\mathsf{T}}$. The system output is defined as $y = [p, v, b_1, b_2]^{\mathsf{T}}$ where b_2 is the signal obtained using an accelerometer, $b_1 = Rr_1$ is a signal obtained using a magnetometer, and r_1 is the magnetic field of the surrounding environment (assumed constant). Note that the system attitude Q, R is not assumed to be a known output of the system. The accelerometer model is given by

$$b_2 = R\,(\dot{v} - ge_3) = -u_t e_3 + R\delta = Rr_2, \tag{12}$$

where r_2 is known as the system *apparent acceleration* such that $\dot{v} = ge_3 + r_2$, or equivalently

$$r_2 = -u_t R^{\mathsf{T}} e_3 + \delta. \tag{13}$$

III. Problem Formulation

Let p_r denote a desired reference position, which is assumed to be constant (or slowly-varying), and let $e_p = p - p_r$. Our main objective is to develop a control law for the system inputs u_t and ω, using the available system outputs $y = [p, v, b_1, b_2]$, such that the system states (e_p, v) are bounded and $\lim_{t\to\infty}(e_p, v) = 0$. To develop the control laws we first make a number of simplifying assumptions.

***Assumption** 1 (Aerodynamic Forces):* In light of the fact that the disturbance force δ is due to aerodynamic forces exerted on the vehicle we make the following simplifying assumptions:

(a) The aerodynamic disturbance δ is dissipative with respect to the system translational kinetic energy and satisfies $\delta^{\mathsf{T}} v \le 0$.
(b) The aerodynamic disturbance force δ is only dependant on the system translational velocity, and there exists a positive constant c_1 such that δ is bounded by $\|\delta\| \le c_1\|v\|^2$.
(c) There exists positive constants c_2 and c_3 such that the time-derivative of the aerodynamic disturbance force is bounded by $\|\dot{\delta}\| < c_2 + c_3\|v\|^3$.

Assumption 1(a) and 1(b) can be realized when the system is operating in an environment where the exogenous airflow is negligible (no wind). Assumption 1(c) can be satisfied when the system geometry is sufficiently symmetrical such that the system aerodynamic forces do not significantly depend on the system orientation. Although this assumption may be reasonable for certain VTOL type aircraft, for example the ducted-fan, this assumption may not be the case with certain systems, for example fixed wing aircraft, where the system aerodynamics depend largely on the orientation of the vehicle.

***Assumption** 2:* Given two positive constants γ_1, γ_2, there exists a positive constant $c_w(\gamma_1, \gamma_2)$ such that the matrix

$$W = -\gamma_1 S(r_1)^2 - \gamma_2 S(r_2)^2, \tag{14}$$

is bounded by $c_w \le \lambda_{\min}(W)$, where $\lambda_{\min}(W)$ denotes the minimum eigenvalue of W.

Assumption 2 is satisfied provided that the value for r_2 is non-vanishing and is non-collinear to the magnetic field vector r_1. Note that in the case where $r_2 = 0$, $\dot{v} = ge_3$ which is unlikely under normal operating conditions.

Now that we have established the required assumptions, let us consider the model for the system acceleration from (10): due to the underactuated nature of this system, the translational acceleration is driven by the system thrust and orientation $\mu(u_t, R)$. That is, if μ were a control input, setting $\mu = -k_p e_p - k_v v$ would satisfy the objectives (since $v^{\mathsf{T}}\delta(v) \le 0$). However, since μ is a function of the system state, we define the $\mu_d \in \mathbb{R}^3$ as the *desired acceleration*, and introduce the new error signal

$$e_\mu = \mu - \mu_d. \tag{15}$$

Subsequently, a new objective is to force $e_\mu \to 0$ in order to obtain the desired translational dynamics. Since the signal μ is dependant on the system thrust and attitude, based upon the value of the desired acceleration μ_d we wish to obtain a suitable *desired attitude*, denoted as $Q_d = (\eta_d, q_d) \in \mathbb{Q}$, and system thrust u_t, such that the following equation is satisfied

$$\mu_d = ge_3 - u_t R_d^{\mathsf{T}} e_3, \tag{16}$$

where $R_d = R\,(\eta_d, q_d)$, is the rotation matrix corresponding to the unit-quaternion Q_d, as defined by (3). An extraction method which satisfies these requirements has been previously given in [8]. For the sake of clarity, this attitude and thrust extraction algorithm is restated in the following section.

A. Desired Attitude and Thrust Extraction

Given an arbitrary vector $\mu_d \in \mathbb{R}^3$ we wish to find the value of attitude $R_d \in SO(3)$ and system thrust $u_t \in \mathbb{R}$ that satisfies

$$ge_3 - u_t R_d^{\mathsf{T}} e_3 = \mu_d. \tag{17}$$

A solution to this problem, which has been proposed in [8], is provided as follows: Given μ_d where $\mu_d \notin L$,

$$L = \{\mu_d \in \mathbb{R}^3; \mu_d = \mathrm{col}[0, 0, \mu_{d3}]; \mu_{d3} \in [g, \infty)\}, \tag{18}$$

then a value of the thrust u_t and attitude $Q_d = (\eta_d, q_d)$ which satisfies (17) is given by

$$u_t = \|\mu_d - ge_3\|, \tag{19}$$

$$\eta_d = \left(\frac{1}{2}\left(1 + \frac{g - e_3^{\mathsf{T}}\mu_d}{\|\mu_d - ge_3\|}\right)\right)^{1/2}, \tag{20}$$

$$q_d = \frac{1}{2\|\mu_d - ge_3\|\eta_d} S(\mu_d) e_3. \tag{21}$$

The extracted attitude Q_d has the time derivative

$$\dot{Q}_d = \frac{1}{2}\begin{bmatrix} -q_d^{\mathsf{T}} \\ \eta_d I_{3\times 3} + S(q_d) \end{bmatrix} \omega_d, \tag{22}$$

where the *desired angular velocity* ω_d is given by

$$\omega_d = M(\mu_d)\dot{\mu}_d, \quad (23)$$

$$M(\mu_d) = \frac{1}{4\eta_d^2 u_t^4}\left(-4S(\mu_d)e_3e_3^{\mathsf{T}} + 4\eta_d^2 u_t S(e_3) + 2S(\mu_d) - 2e_3^{\mathsf{T}}\mu_d S(e_3)\right) S\left(\mu_d - ge_3\right)^2. \quad (24)$$

It is worth noting that there are an infinite number of solutions for the desired attitude which achieve the desired system acceleration. The above attitude extraction offers a unique solution since it effectively includes an extra constraint by fixing the desired yaw angle.

B. Attitude Error

Let $Q_e = (\eta_e, q_e) \in \mathbb{Q}$ and $R_e = R(\eta_e, q_e) \in SO(3)$ denote the *attitude error* which are defined by $Q_e = Q \odot Q_d^{-1}$, and $R_e = R_d^{\mathsf{T}} R$, where Q_d is the unit quaternion obtained using (20) and (21). In light of the derivatives $\dot{Q}$ and $\dot{Q}_d$, as defined by (11) and (22), respectively, the time derivative of the attitude error is found to be

$$\dot{Q}_e = \frac{1}{2}Q_e \odot \left[\begin{array}{c} 0 \\ \omega_e \end{array}\right], \; \dot{R}_e = -S(\omega_e)R_e, \; \omega_e = R_d^{\mathsf{T}}\left(\omega - \omega_d\right), \quad (25)$$

where ω_d is the *desired angular velocity* as defined by (23).

IV. Position Controller

As previously discussed, the control scheme is based upon a value of the desired attitude, in terms of R_d or Q_d, which is obtained using the value of the desired translational acceleration μ_d. To avoid the singularity in the attitude extraction algorithm, defined by (18), we must ensure that the expression for μ_d is bounded a priori. This can be accomplished using the bounded function $h(\cdot)$ defined in section (II-C). We choose the following expression for the desired translational acceleration:

$$\mu_d = -k_p h(e_p) - k_v h(v), \quad (26)$$

from which we obtain the required system thrust u_t and desired attitude $Q_d = (\eta_d, q_d)$ from (19)-(21), and the rotation matrix R_d using (3). To avoid the singularity in the attitude extraction (to ensure a solution for Q_d and R_d exists) we choose positive gains k_p and k_v which satisfy the following requirement

$$k_p + k_v < g, \quad (27)$$

which ensures the singularity can never be reached. Due to this choice of gains, from (19) one can deduce that the thrust is non-vanishing and bounded. That is there exists positive constants $\underline{c}_t$ and $\bar{c}_t$ such that

$$0 < \underline{c}_t \leq u_t \leq \bar{c}_t, \qquad \underline{c}_t = g - k_p - k_v, \quad \bar{c}_t = g + k_p + k_v. \quad (28)$$

Furthermore, as a consequence of the lower bound for the thrust, in [8] the authors find an upper bound for the matrix $M(\mu_d)$, defined by (24), which is given by

$$\|M(\mu_d)\| \leq \sqrt{2}/\underline{c}_t. \quad (29)$$

In light of the choice for μ_d, the derivatives of the position error and velocity are now given by

$$\dot{e}_p = v, \quad (30)$$

$$\dot{v} = -k_p h(e_p) - k_v h(v) + e_\mu + \delta. \quad (31)$$

To obtain the desired angular velocity ω_d, using (8), (23), (30) and (31) we first calculate the derivative of μ_d to be

$$\dot{\mu}_d = f_{\mu_d} - k_v\phi(v)e_\mu - k_v\phi(v)\delta, \quad (32)$$

$$f_{\mu_d} = -k_p\phi(e_p)v + k_v\phi(v)\left(k_p h(e_p) + k_v h(v)\right). \quad (33)$$

To define the control law, we are required to use vector measurements (instead of the attitude Q or R), however the inertial referenced vector r_2 which corresponds to the accelerometer measurement is not available. To obviate the requirement of r_2, we define the error signal $\tilde{v} = v - \hat{v}$ where $\hat{v} \in \mathbb{R}^3$ is an adaptive state which is designed such that $\tilde{v}$ is related to r_2 in some manner. To this end, we now propose the following control law

$$\omega = M(\mu_d)\left(f_{\mu_d} - k_v\phi(v)R_d^{\mathsf{T}}\left(b_2 + u_t e_3\right)\right) + \beta, \quad (34)$$

$$\beta = \gamma_1 S(R_d r_1)b_1 + \gamma_2 k_1 S\left(R_d\left(v - \hat{v}\right)\right)b_2 \quad (35)$$

$$\dot{\hat{v}} = ge_3 + R_d^{\mathsf{T}}b_2 + k_1\left(v - \hat{v}\right) + R_d^{\mathsf{T}}S(b_2)\beta/k_1, \quad (36)$$

where $k_1, \gamma_1, \gamma_2 > 0$, $M(\mu_d)$ is the function defined by (24), and $\phi(\cdot)$ is the bounded function defined by (8). The role of β in the feedback structure is to use the vector measurements in order to force the system attitude to the desired value (correction term which forces $Q \to Q_d$). The remaining term in the expression for ω is related to the desired angular velocity ω_d (tracking term).

In light of these control laws we propose the following theorem:

***Theorem** 1: Consider the system given by (9)-(11), where we apply the control laws u_t and ω as defined by (19) and (34) where the gain restriction (27) is satisfied. Let Assumptions 1 and 2 be satisfied. Then the system thrust u_t is bounded and non-vanishing, and for all initial conditions $\eta_e(t_0) \neq 0$ there exists gains $\bar{\gamma}_1, \bar{\gamma}_2, \kappa_1 > 0$ such that for $\gamma_1 > \bar{\gamma}_1$, $\gamma_2 > \bar{\gamma}_2$, $k_1 > \kappa_1$, the system states (e_p, v) are bounded and $\lim_{t\to\infty}(e_p, v) = \mathbf{0}$.*

Proof: First, let us define the following error function

$$\psi = k_1\tilde{v} - (I - R_e)\, r_2,$$

where $\tilde{v} = v - \hat{v}$ and $I = I_{3\times 3}$ is the identity matrix. Using the error function for ψ, in addition to the fact that $S(R_d u) = R_d S(u) R_d^{\mathsf{T}}$ and $R_d^{\mathsf{T}} R = R_e$, the expression for the function β, defined by (35), can now be written as

$$\beta = \gamma_1 R_d S(r_1)R_e r_1 + \gamma_2 R_d S(r_2)R_e r_2 + \gamma_2 R_d S(\psi) R_e r_2. \quad (37)$$

At this point in the proof, we wish to study the dynamics of the attitude error (in terms of the quaternion scalar η_e) as a result of the proposed control law, which requires we first calculate the expression for the desired angular velocity ω_d. Due to the control law μ_d from (26), the derivative $\dot{\mu}_d$ from (32), and using (23) we find the expression for the desired angular velocity is given by $\omega_d = M(\mu_d)\left(f_{\mu_d} - k_v\phi(v)\delta - k_v\phi(v)e_\mu\right)$, which we use, along with (25), (37), the control law ω from

(34), and the properties $b_2 + u_t e_3 = R\delta$ and $q_e^\mathsf{T} S(r_i) R_e r_i = 2\eta_e q_e^\mathsf{T} S(r_i)^2 q_e$. to eventually obtain the time-derivative of η_e to be given by

$$\begin{aligned}\dot{\eta}_e \;&=\; \eta_e q_e^\mathsf{T} W q_e - \frac{\gamma_2}{2} q_e^\mathsf{T} S(\psi) R_e r_2 \\ &\quad - \frac{k_v}{2} q_e^\mathsf{T} R_d^\mathsf{T} M(\mu_d)\phi(v)\left((I - R_e)\,\delta + e_\mu\right), \end{aligned} \quad (38)$$

where W is the matrix defined by (14). Note that due to Assumption 2, the matrix $W = W^\mathsf{T}$ is positive-definite. We now focus our attention to study the dynamics of the error function ψ. In light of the expression for $\dot{v}$ from (10), the attitude error dynamics (25), the expression for $\dot{\hat{v}}$ from (36), and using the fact that $-k_1\tilde{v} + r_2 - \hat{R}^\mathsf{T} b_2 = -\psi$, we obtain

$$\begin{aligned}\dot{\psi} \;&=\; -k_1\psi - (I - R_e)\,\dot{r}_2 \\ &\quad + k_v R_d^\mathsf{T} S(b_2) M(\mu_d)\phi(v)\left((I - R_e)\,\delta + e_\mu\right), \end{aligned} \quad (39)$$

For the sake of convenience, we define two functions, $f_1(u_t), f_2(x) \in \mathbb{R}^{3\times 3}$ such that

$$e_\mu = f_1(u_t) q_e, \quad (I - R_e)x = f_2(x) q_e, \quad x \in \mathbb{R}^3,$$

where from the definition of $e_\mu = \mu - \mu_d$, in addition to the expressions for μ and μ_d from (10) and (17), respectively, one can find $f_1(u_t) = 2u_t\left(\eta_e I - S(q_e)\right) S(R^\mathsf{T} e_3)$ and $f_2(x) = 2\left(S(q_e) - \eta_e I\right) S(x)$. Based upon these definitions we find the following upper bounds for these two functions

$$\|f_1(u_t)\| \le 2\bar{c}_t \qquad \|f_2(x)\| \le 2\|x\|. \quad (40)$$

We now propose the following Lyapunov function candidate:

$$\mathcal{V} = \gamma k_p\left(\sqrt{1 + e_p^\mathsf{T} e_p} - 1\right) + \frac{\gamma}{2} v^\mathsf{T} v + \frac{\gamma k_\psi}{2}\psi^\mathsf{T}\psi + \gamma_q\left(1 - \eta_e^2\right), \quad (41)$$

where $\gamma, k_\psi, \gamma_q > 0$. In light of (30), (31), (38), (39), and the expression for μ_d defined by (26), we derive the derivative of $\mathcal{V}$ as follows:

$$\begin{aligned}\dot{\mathcal{V}} &= -\gamma k_v v^\mathsf{T} h(v) + \gamma v^\mathsf{T}\delta - \gamma k_\psi k_1 \psi^\mathsf{T}\psi - 2\gamma_q \eta_e^2 q_e^\mathsf{T} W q_e \\ &\quad + \gamma k_v k_\psi \psi^\mathsf{T} R_d^\mathsf{T} S(b_2) M(\mu_d)\phi(v)\left(f_1(u_t) + f_2(\delta)\right) q_e \\ &\quad - \gamma k_\psi \psi^\mathsf{T} f_2(\dot{r}_2) q_e + \gamma v^\mathsf{T} f_1(u_t) q_e + \gamma_2\gamma_q\eta_e q_e^\mathsf{T} S(\psi) R_e r_2 \\ &\quad + \gamma_q k_v \eta_e q_e^\mathsf{T} R_d^\mathsf{T} M(\mu_d)\phi(v)\left(f_1(u_t) + f_2(\delta)\right) q_e. \end{aligned} \quad (42)$$

To study the upper bound for $\dot{\mathcal{V}}$ we first study the bounds for a number of functions. First, let us define the function $\sigma(t) \in \mathbb{R}$ where

$$\sigma(t) := \sqrt{2\mathcal{V}(t)}. \quad (43)$$

Based upon the definition of $\mathcal{V}$ from (41), one can find $\|v(t)\| \le \sigma(t)/\sqrt{\gamma}$ and $\|\psi(t)\| \le \sigma(t)/\sqrt{\gamma k_\psi}$. Therefore, in light of Assumption 1, one can conclude that the value of $\delta(v)$ is bounded by

$$\|\delta(v)\| \le c_1\sigma(t)^2/\gamma. \quad (44)$$

Due to the bounds of the functions $f_1(u_t)$ and $f_2(\delta)$ from (40), and the definition of r_2 from (13) we also find

$$\|f_1(u_t) + f_2(\delta)\| \le 2\left(\gamma\bar{c}_t + c_1\sigma(t)^2\right)/\gamma, \quad (45)$$

$$\|b_2\| \le \left(\gamma\bar{c}_t + c_1\sigma(t)^2\right)/\gamma, \quad (46)$$

Given these bounds, we now apply Young's inequality to a number of the undesired terms in the expression for $\dot{\mathcal{V}}$:

$$\gamma v^\mathsf{T} f_1(u_t) q_e \le \frac{\gamma\epsilon_1}{2} v^\mathsf{T} h(v) + \frac{2\sqrt{\gamma}\bar{c}_t^2}{\epsilon_1}\sqrt{\gamma + \sigma(t)^2}\, q_e^\mathsf{T} q_e, \quad (47)$$

$$\begin{aligned}&\gamma k_v k_\psi \psi^\mathsf{T} R_d^\mathsf{T} S(b_2) M(\mu_d)\phi(v)\left(f_1(u_t) + f_2(\delta)\right) q_e \\ &\le \frac{\gamma k_v k_\psi \epsilon_2}{2}\psi^\mathsf{T}\psi + \frac{4k_v k_\psi}{\epsilon_2\gamma^3\underline{c}_t^2}\left(\gamma\bar{c}_t + c_1\sigma(t)^2\right)^4 q_e^\mathsf{T} q_e, \end{aligned} \quad (48)$$

where the norm of the matrix $M(\mu_d)$ is given by (29). To determine the bound of the term involving the derivative of r_2, we differentiate the expression for r_2 given by (13), which we have omitted due to space constraints. However, based upon the result for the derivative, in addition to the bound of b_2 and δ from (46) and (44), respectively, we find that there exists positive constants $d_{1,2,3,4,5} > 0$, such that the norm of $\dot{r}_2$ is bounded by $\dot{r}_2 \le d_1 + d_2\|v\| + d_3\|v\|^2 + d_4\|v\|^3 + d_5\|v\|^4$. However, for the sake of simplicity, from this result we further conclude that there exists positive constants c_3 and c_4 such that $\dot{r}_2 \le c_3 + c_4\sigma(t)^4$. Note that the constants d_i, and therefore c_3 and c_4, depend on the gains k_p, k_v, γ_1 and γ_2, but they do not depend on the gain k_1, which will be useful later in the proof. As a result of this analysis, we again use Young's inequality to establish the following bounds:

$$\begin{aligned}&\gamma k_\psi \psi^\mathsf{T} f_2(\dot{r}_2) q_e \\ &\quad \le \frac{\gamma k_\psi \epsilon_3}{2}\psi^\mathsf{T}\psi + \frac{2\gamma k_\psi}{\epsilon_3}\left(c_3 + c_4\sigma(t)^4\right)^2 q_e^\mathsf{T} q_e, \end{aligned} \quad (49)$$

$$\begin{aligned}&\gamma_2\gamma_q\eta_e q_e^\mathsf{T} S(\psi) R_e r_2 \\ &\quad \le \frac{\gamma_2\gamma_q\epsilon_4}{2}\psi^\mathsf{T}\psi + \frac{\gamma_2\gamma_q}{2\gamma^2\epsilon_4}\left(\bar{c}_t\gamma + c_1\sigma(t)^2\right)^2 \eta_e^2 q_e^\mathsf{T} q_e. \end{aligned} \quad (50)$$

Finally, we also find the bound of the following function

$$\begin{aligned}&\gamma_q k_v \eta_e q_e^\mathsf{T} R_d^\mathsf{T} M(\mu_d)\phi(v)\left(f_1(u_t) + f_2(\delta)\right) q_e \\ &\le 2\sqrt{2}\gamma_q k_v\left(\gamma\bar{c}_t + c_1\sigma(t)^2\right)|\eta_e| q_e^\mathsf{T} q_e/(\gamma\underline{c}_t). \end{aligned} \quad (51)$$

Recall from Assumption 2 the norm of the matrix W has a lower bound which is denoted as c_w. Therefore, in light of these results we find the expression $\dot{\mathcal{V}}$ is bounded by

$$\begin{aligned}&\dot{\mathcal{V}}(t) \le -v^\mathsf{T} h(v)\left(k_v - \epsilon_1/2\right) \\ &-\gamma_q\eta_e^2 q_e^\mathsf{T} q_e\Big(2c_w - \frac{1}{\eta_e^2}\left(\frac{\alpha_1(t)}{\epsilon_1} + \frac{\alpha_2(t)}{\epsilon_2} + \frac{\alpha_3(t)}{\epsilon_3}\right) - \frac{\alpha_4(t)}{\epsilon_4|\eta_e|} \\ &-\frac{2\sqrt{\gamma}\bar{c}_t^2\left(\gamma + \sigma(t)^2\right)^{1/2}}{\eta_e^2} - \frac{2\sqrt{2}k_v\left(\gamma\bar{c}_t + c_1\sigma(t)^2\right)}{\gamma\underline{c}_t|\eta_e|}\Big) \\ &-\gamma k_\psi\psi^\mathsf{T}\psi\Big(k_1 - \frac{\epsilon_2 k_v + \epsilon_3}{2} - \frac{\epsilon_4\gamma_2\gamma_q}{2\gamma k_\psi}\Big), \end{aligned} \quad (52)$$

$$\begin{aligned}\alpha_1(t) &= 2\sqrt{\gamma}\bar{c}_t^2\sqrt{\gamma + \sigma(t)^2}/\gamma_q, \\ \alpha_2(t) &= 4k_v k_\psi\left(\gamma\bar{c}_t + c_1\sigma(t)^2\right)^4/(\gamma^3\underline{c}_t^2\gamma_q), \\ \alpha_3(t) &= 2\gamma k_\psi(c_3 + c_4\sigma(t)^4)^2/\gamma_q, \\ \alpha_4(t) &= \gamma_w(\bar{c}_t\gamma + c_1\sigma(t)^2)^2/(2\gamma^2). \end{aligned} \quad (53)$$

Note that when $\eta_e(t) = 0$ we cannot guarantee stability using (52) since in this case $\dot{\mathcal{V}}$ could potentially be positive. Consequently, we must show that $\eta_e(t) \ne 0$ for all $t > t_0$. Due to this requirement we are forced to exclude the initial condition $\eta_e(t_0) = 0$. To show that $\eta_e(t)$ is never zero, we first introduce the positive constant ρ which is the desired minimum bound for $|\eta_e(t)|$. Therefore, ρ must be chosen to satisfy

$0 < \rho < |\eta_e(t_0)|$. Subsequently, based upon the definition of the Lyapunov function candidate (41), we choose the gain γ as follows

$$\gamma = \frac{\bar{\gamma}}{k_p\left(\sqrt{1+\|e_p(t_0)\|^2}-1\right)+\frac{1}{2}\|v(t_0)\|^2+\frac{k_\psi}{2}\|\psi(t_0)\|^2+\xi}, \tag{54}$$

where $\xi > 0$, and $\bar{\gamma}$ is chosen to satisfy

$$0 < \bar{\gamma} < \gamma_q\left(\eta_e(t_0)^2 - \rho^2\right). \tag{55}$$

The remaining gains are chosen as follows: Choose $k_p, k_v > 0$ such that (27) is satisfied, and choose $0 < \epsilon_1 < 2k_v$. Recall from (14) that the gain $c_w > 0$ can be increased using the gains γ_1 and γ_2. Therefore, there exists gains $\bar{\gamma}_1$,$\bar{\gamma}_2$, and $\bar{\epsilon}_i$, $i = 2, 3, 4$, such that for all $\gamma_1 > \bar{\gamma}_1$, $\gamma_2 > \bar{\gamma}_2$, and $\epsilon_i > \bar{\epsilon}_i$ the following inequality is satisfied

$$\begin{aligned}2c_w > &\frac{1}{\rho^2}\left(\frac{\alpha_1(t_0)}{\epsilon_1}+\frac{\alpha_2(t_0)}{\epsilon_2}+\frac{\alpha_3(t_0)}{\epsilon_3}\right)+\frac{\alpha_4(t_0)}{\epsilon_4\rho}\\ &+\frac{2\sqrt{\gamma}\bar{c}_t^2\left(\gamma+\sigma(t_0)^2\right)^{1/2}}{\rho^2}+\frac{2\sqrt{2}k_v\left(\gamma\bar{c}_t+c_1\sigma(t_0)^2\right)}{\gamma\underline{c}_t\rho},\end{aligned} \tag{56}$$

where we note that the functions $\sigma(t)$ and $\alpha_1(t)$ through $\alpha_4(t)$ are non-increasing if $\dot{\mathcal{V}} \le 0$. Finally, we choose the gain k_1 to satisfy $k_1 > \kappa_1 := k_1 > \epsilon_2 k_v + \epsilon_3/2 + \epsilon_4\gamma_2\gamma_q/(2\gamma k_\psi)$. Due to these choices for the gains we conclude that $\dot{\mathcal{V}}(t_0) \le 0$, and a sufficient condition for $\dot{\mathcal{V}}(t) \le 0$ is $|\eta_e(t)| \ge \rho$. We will now show that indeed $\rho \le |\eta_e(t)|$ for all $t > t_0$. Suppose that there exists a time t_1 such that for all $t_0 \le t < t_1$, $|\eta_e(t)| \ge \rho$ and $|\eta_e(t_1)| < \rho$ when $t = t_1$. At the time t_1 from (41), it is clear that $\mathcal{V}(t_1) \ge \gamma_q\left(1-\eta_e(t_1)^2\right) > \gamma_q\left(1-\rho^2\right)$. However, due to the choice of the gain γ and $\bar{\gamma}$, given by (54) and (55), respectively, the value of the Lyapunov function candidate at the initial time t_0 must satisfy $\mathcal{V}(t_0) < \bar{\gamma}+\gamma_q\left(1-\eta_e(t_0)^2\right) < \gamma_q\left(1-\rho^2\right)$ and therefore $\mathcal{V}(t_1) > \mathcal{V}(t_0)$. This is a contradiction since $\dot{\mathcal{V}}(t) \le 0$ for all $t_0 \le t < t_1$. Therefore, we conclude that $|\eta_e(t)| \ge \rho$ and $\dot{\mathcal{V}}(t) \le 0$ for all $t > t_0$, and the states (v, ψ) are bounded. Therefore the expressions for $\dot{\psi}$, $\dot{v}$, $\dot{\eta}_e$, and $\ddot{\mathcal{V}}$ are bounded. Barbalat's Lemma therefore implies that $\lim_{t\to\infty}(v, \psi, q_e) = 0$. Since $\psi \to 0$ and $q_e \to 0$, this further implies that $\tilde{v} \to 0$. Furthermore, since $\lim_{t\to\infty}\dot{v} = 0$, and $\lim_{t\to\infty}\delta = 0$, it follows from the expression of the velocity dynamics $\dot{v} = -k_p h(e_p) - k_v h(v) - \delta = 0$, then $\lim_{t\to\infty} e_p = 0$ which satisfies the control objectives. ■

V. Conclusion

We proposed a new position controller for VTOL-UAVs which uses low cost sensors, namely an accelerometer and magnetometer, in addition to a GPS. The main advantage of this controller is that it does not require direct measurement of the system attitude, nor does it require the use of an attitude observer. Alternatively, in this control-scheme we use the accelerometer and magnetometer signals directly in the position controller (and not for attitude estimation). Furthermore, the accelerometer is now used to obtain the system apparent acceleration, instead of measuring the gravity vector in the body-fixed frame (which would normally require the system to be non-accelerating), which offers superior performance when the system is subjected to linear accelerations. We have shown that, through an appropriate choice of the control gains, the system position is guaranteed to converge to the desired position for almost all initial conditions.

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