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Consensus algorithms design for constrained heterogeneous multi-agent systems

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Abstract— **This paper addresses the output consensus problem of heterogeneous multi-agent systems. Specifically, we propose consensus algorithms that account for input saturations using only the available output measurements. Our approach takes roots from a simple design procedure built around *ideal* first order consensus schemes (full state availability and no input saturations) and extends to the more challenging case including systems heterogeneity, input saturations and partial state availability. Numerical examples are provided to illustrate the effectiveness of the obtained theoretical results.**

I. Introduction

The consensus problem in multi-agent systems has been actively studied over the last decade due to its numerous applications in the cooperative control of multiple vehicle systems. Using local information exchange, the consensus problem consists in finding appropriate algorithms that drive a team of agents to reach an agreement on some consistent quantities or a common objective. Using graph theory, matrix theory, frequency-domain analysis tools and Lyapunov methods, several consensus algorithms for multi-agent systems with first order dynamics have been proposed in the literature, see for example [1]-[8] and references therein. The consensus problem of multi-agent systems with second order dynamics has also received a wide interest, especially in the last few years, leading to several interesting results such as [9]-[13] to cite only a few. This interest is motivated by the close relationship between consensus problems and motion coordination of complex dynamical systems such as formation control, Rendez-vous, and synchronization problems.

Although a particular emphasis in the above references is given to the study of the effects of the communication topology between agents, several practical constraints that are generally imposed in practical environments have not been considered. In multiple vehicles applications for example, the inputs are generally subject to input saturations. Also, in several applications, it is desirable to design controllers that do not involve the measurements of some state variables. Using only partial state measurements, some consensus algorithms have been proposed for second order multi-agent systems using lead filters in [11] and for agents with higher order using reduced order observers in [14]. Also, second order consensus algorithms that account for input saturations have been proposed in [11] and [15] in the case of undirected communication topologies. However, the design of consensus algorithms that take into account these two constraints simultaneously is a challenging problem. In [16], a solution to this problem for second order multi-agent systems has been proposed in the simple case of undirected communication topologies. The main idea in [16] is based on the use of some auxiliary dynamics that alter the trajectories of the agents before reaching consensus. Based on this idea, it has been shown in [17] that, under some assumptions, existing second-order consensus algorithms developed in the full state information case and without input constraints can be adapted to solve the velocity-free second-order consensus algorithms and account for input saturations.

While most of the aforementioned papers have been focused on homogeneous multi-agent systems, *i.e.,* agents with identical dynamics, some efforts have been recently made to solve the consensus problem of heterogeneous multi-agent systems, which is important in applications involving the coordination of different types of mobile agents. In [18], a solution to the consensus problem for a class of uncertain heterogeneous linear multi-agent systems is proposed, where the internal model principle [19] is used to design appropriate consensus algorithms that drive all agents to achieve output consensus. Also based on the internal model principle, the authors in [20] derive necessary and sufficient conditions such that output consensus is achieved in heterogeneous linear multi-agent systems. Despite the interesting results cited above, the design of output consensus algorithms for heterogeneous multi-agent systems in the presence of input saturation constraints is yet a challenging and difficult problem.

In this paper, we present consensus algorithms for heterogeneous multi-agent systems that account for input saturations using only the available outputs. We focus on multi-agent systems containing agents with first order and second order dynamics, and propose a unified approach to the consensus algorithm design problem in this case. In the spirit of [16] and [17], our approach is based on the introduction of dynamic auxiliary systems to direct the agents towards suitable intermediate trajectories before reaching consensus. Our approach can also be interpreted using the internal model principle as done in [18] and [20]. In contrast to the latter papers, the order of the auxiliary dynamic system depends on the dynamics of the agent and two different intermediate reference trajectories are defined for each agent to achieve output consensus for the heterogeneous multi-agent system and account for input saturations. Sufficient conditions are

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derived such that existent consensus algorithms developed for single-integrator multi-agent systems can be applied to heterogeneous multi-agents in this case. The application and effectiveness of the proposed design method are illustrated by two numerical examples, where we provide solutions to the consensus problem with communication delays and the consensus problem with a common desired velocity.

II. Problem description

Consider the heterogeneous multi-agent system composed of n autonomous agents, labeled 1 through n, where the first μ agents, with $\mu < n$, are governed by second order dynamics and the remaining $(n-\mu)$ agents are modeled by first order dynamics, *i.e.,*

$$\begin{array}{rcll} \ddot{\mathbf{p}}_i & = & \mathbf{u}_i, & \text{for } i \in \mathcal{N}_1, \\ \dot{\mathbf{p}}_i & = & \mathbf{u}_i, & \text{for } i \in \mathcal{N}_2, \end{array} \tag{1}$$

where $\mathcal{N}_1 \triangleq \{1,...,\mu\}$, $\mathcal{N}_2 := \{\mu+1,...,n\}$, $\mathbf{p}_i \in \mathbb{R}^m$ and $\dot{\mathbf{p}}_i$ denote respectively the position-like and velocity-like states of the i^{th} agent and the vector $\mathbf{u}_i \in \mathbb{R}^m$ is the control input. Note that $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$.

The information exchange between agents is represented by a weighted graph $\mathcal{G}_n = (\mathcal{N}, \mathcal{E}, \mathcal{K})$, where $\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2$ is the set of nodes or vertices, describing the set of all agents in the multi-agent system, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of pairs of nodes, called edges, and $\mathcal{K} = [k_{ij}]$ is a weighted adjacency matrix. An edge $(i,j) \in \mathcal{E}$ indicates that agent i can receive information from agent j, which is designated as its neighbor. The weighted adjacency matrix of a weighted graph is defined such that $k_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$ and $k_{ij} = 0$ if and only if $(i,j) \notin \mathcal{E}$. If the communication topology is bidirectional, then $\mathcal{G}_n$ is undirected, the pairs of nodes in $\mathcal{E}$ are unordered, $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$, and $\mathcal{K}$ is symmetric. In the case of unidirectional communication topology, $\mathcal{G}_n$ is a directed graph, $\mathcal{E}$ contains ordered pairs, and $\mathcal{K}$ is not necessarily symmetric. In the case where the communication topology is dynamically changing, due to restrictions imposed by the environment for example, the weights k_{ij} are time-varying. Also, the information exchange between agents in the team can be subject to communication delays.

We assume that all agents are subject to input saturations, such that $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$, for $i \in \mathcal{N}$, and the velocity vectors of the μ first agents, *i.e.,* agents with second order dynamics, are not available for feedback. In the presence of these two constraints, the objective of our work is to present consensus algorithms for the heterogeneous multi-agent system (1), under a certain communication topology described by $\mathcal{G}_n$, such that [1]

$$(\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_d) \to 0, \quad (\mathbf{p}_i - \mathbf{p}_j) \to 0, \quad \text{for } i,j \in \mathcal{N}, \tag{2}$$

for any initial conditions, with $\dot{\mathbf{p}}_d$ being a desired velocity available to all members of the team, which can be time-varying, constant or null, and satisfies $\|\dot{\mathbf{p}}_d\|_\infty \leq \mathbf{v}_{\max} < \mathbf{u}_{\max}$ and $\|\ddot{\mathbf{p}}_d\|_\infty \leq \mathbf{a}_{\max} < \mathbf{u}_{\max}$.

For a given vector $\mathbf{x} = (x^1,...,x^m)^\top \in \mathbb{R}^m$, we define the saturation function

$$\boldsymbol{\chi}(\mathbf{x}) = \mathrm{col}[\sigma(x^k)] \in \mathbb{R}^m, \quad \text{for } k \in \{1,...,m\}, \tag{3}$$

with $\sigma : \mathbb{R} \to \mathbb{R}$, being a strictly increasing continuously differentiable function satisfying the following properties:

P1. $\sigma(0) = 0$ and $x\sigma(x) > 0$ for $x \neq 0$,

P2. $|\sigma(x)| \leq \sigma_b$, for $\sigma_b > 0$.

P3. The diagonal matrix $\mathbf{h}(\mathbf{x}) = \mathrm{diag}[\frac{\partial \sigma(x^k)}{\partial x^k}]$ satisfies $\|\mathbf{h}(\mathbf{x})\| \leq \sigma_h$, $\sigma_h > 0$.

III. Main result

In this section, we present a unified approach to the consensus algorithm design for the heterogeneous multi-agent system (1) under the communication topology described by $\mathcal{G}_n$. Let us associate to each agent the following dynamic systems

$$\dot{\bar{\mathbf{r}}}_i = \dot{\mathbf{p}}_d - L_i^\zeta \boldsymbol{\chi}(\boldsymbol{\zeta}_i), \quad \text{for } i \in \mathcal{N}_1, \tag{4}$$

and

$$\dot{\bar{\mathbf{r}}}_i = \dot{\mathbf{p}}_d - L_i^\xi \boldsymbol{\chi}(\boldsymbol{\xi}_i), \quad \text{for } i \in \mathcal{N}_2, \tag{5}$$

where L_i^ζ and L_i^ξ are strictly positive scalar gains, $\bar{\mathbf{r}}_i \in \mathbb{R}^m$ can take arbitrary initial conditions, the function $\boldsymbol{\chi}$ is defined in (3), and the vectors $\boldsymbol{\zeta}_i \in \mathbb{R}^m$ and $\boldsymbol{\xi}_i \in \mathbb{R}^m$ obey to the following dynamics

$$\dot{\boldsymbol{\zeta}}_i = -L_i^\zeta \boldsymbol{\chi}(\boldsymbol{\zeta}_i) + L_i^\xi \boldsymbol{\chi}(\boldsymbol{\xi}_i), \quad \text{for } i \in \mathcal{N}_1, \tag{6}$$

$$\dot{\boldsymbol{\xi}}_i = -L_i^\xi \boldsymbol{\chi}(\boldsymbol{\xi}_i) - \boldsymbol{\Phi}_{i,\mathcal{G}_n}(\hat{\mathbf{r}}), \quad \text{for } i \in \mathcal{N}, \tag{7}$$

and can be initialized arbitrarily, where $\hat{\mathbf{r}} = (\mathbf{r}_1^\top, \dots, \mathbf{r}_n^\top)^\top \in \mathbb{R}^{nm}$, the vector $\mathbf{r}_i \in \mathbb{R}^m$ is defined as

$$\mathbf{r}_i = \begin{cases} \bar{\mathbf{r}}_i - \boldsymbol{\zeta}_i - \boldsymbol{\xi}_i, & \text{for } i \in \mathcal{N}_1, \\ \bar{\mathbf{r}}_i - \boldsymbol{\xi}_i, & \text{for } i \in \mathcal{N}_2, \end{cases} \tag{8}$$

and $\boldsymbol{\Phi}_{i,\mathcal{G}_n}(\hat{\mathbf{r}})$ is a protocol designed using the states $\hat{\mathbf{r}}$, and satisfies the following condition.

Design Condition 1: The multi-agent system

$$\dot{\mathbf{r}}_i = \dot{\mathbf{p}}_d + \boldsymbol{\Phi}_{i,\mathcal{G}_n}(\hat{\mathbf{r}}), \quad \text{for } i \in \mathcal{N}. \tag{9}$$

achieves first order consensus, *i.e.,* $\mathbf{r}_i \to \mathbf{p}_d$ and $(\mathbf{r}_i - \mathbf{r}_j) \to 0$ as $t \to +\infty$, for all $i,j \in \mathcal{N}$, where $\mathcal{G}_n$ is the weighted graph representing the communication topology between agents that can be restricted to be directed, time-varying, and/or subject to communication delays. Further, the solutions of (9) guarantee that $\boldsymbol{\Phi}_{i,\mathcal{G}_n}(\hat{\mathbf{r}})$ is globally bounded and converges to zero when the multi-agent system (9) achieves first-order consensus.

It should be noted that the dynamics (9) describe a multi-agent system with identical agents governed by single-integrator dynamics, where the states $\hat{\mathbf{r}}$ are available for feedback and no constraints are imposed on the right-hand side of (9). Therefore, design condition 1 can be satisfied

[1] Throughout the paper, we omit the arguments of time-dependent signals except for those that are delayed, and use the notation $y \to c$, for a constant c, to indicate that $\lim_{t\to\infty} y(t) = c$.

if one is able to design a consensus algorithm for a multi-agent system with single-integrator dynamics with no input saturation constraints.

With the above definitions, we propose the following control input

$$\mathbf{u}_i = \ddot{\mathbf{p}}_d - k_i^d \boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i) - k_i^p \boldsymbol{\chi}(\mathbf{e}_i) - L_i^\zeta \mathbf{h}(\boldsymbol{\zeta}_i)\left(-L_i^\zeta \boldsymbol{\chi}(\boldsymbol{\zeta}_i) + L_i^\xi \boldsymbol{\chi}(\boldsymbol{\xi}_i)\right), \quad (10)$$

$$\dot{\boldsymbol{\psi}}_i = k_i^\psi(\mathbf{e}_i - \boldsymbol{\psi}_i), \quad (11)$$

for $i \in \mathcal{N}_1$, and

$$\mathbf{u}_i = \dot{\mathbf{p}}_d - L_i^\xi \chi(\boldsymbol{\xi}_i) - k_i^p \boldsymbol{\chi}(\mathbf{e}_i), \quad \text{for} \quad i \in \mathcal{N}_2, \quad (12)$$

where $\mathbf{e}_i = (\mathbf{p}_i - \bar{\mathbf{r}}_i)$, k_i^p, k_i^d, and k_i^ψ are strictly positive scalar gains, $\boldsymbol{\psi}_i(0)$ can take arbitrary initial values, and $\bar{\mathbf{r}}_i$, $\boldsymbol{\zeta}_i$, and $\boldsymbol{\xi}_i$ are obtained from (4)-(7). The function χ is defined in (3), and the diagonal matrix $\mathbf{h}(\cdot)$ is defined in property P3. We can verify, using properties P2 and P3, that the above control input can be upper bounded as

$$\|\mathbf{u}_i\|_\infty \leq \alpha_i(\mathbf{a}_{\max} + \sigma_h \sigma_b L_i^\zeta (L_i^\zeta + L_i^\xi) + \sigma_b k_i^d) + \sigma_b k_i^p + (1 - \alpha_i)(\mathbf{v}_{\max} + \sigma_b L_i^\xi), \quad (13)$$

with $\alpha_i = 1$ for $i \in \mathcal{N}_1$, and $\alpha_i = 0$ for $i \in \mathcal{N}_2$. Note that the right hand side of (13) is always positive due to the definition of α_i and the assumptions on $\dot{\mathbf{p}}_d$ and $\ddot{\mathbf{p}}_d$.

Our main result is given in the following theorem.

Theorem 1: Consider the heterogeneous multi-agent system (1) with a communication topology described by $\mathcal{G}_n$. Let the control input in (1) be given as (10)-(12) with (4)-(8) and suppose that design condition 1 is satisfied. If the control gains are selected such that

$$\gamma_i \leq \mathbf{u}_{\max} - \alpha_i \mathbf{a}_{\max} - (1 - \alpha_i)\mathbf{v}_{\max}, \quad (14)$$

with $\gamma_i = \sigma_b \left(k_i^p + \alpha_i(k_i^d + \sigma_h L_i^\zeta (L_i^\zeta + L_i^\xi)) + (1 - \alpha_i) L_i^\xi\right)$ and α_i being defined after (13), then $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$, for $i \in \mathcal{N}$, and the heterogeneous multi-agent system (1) achieves consensus in the sense of (2), *i.e.,* $(\mathbf{p}_i(t) - \mathbf{p}_j(t)) \to 0$, $(\dot{\mathbf{p}}_i(t) - \dot{\mathbf{p}}_d(t)) \to 0$, as $t \to \infty$, for $i, j \in \mathcal{N}$.

Proof: See Appendix I. ■

The proposed consensus algorithm in Theorem 1 is based on the introduction of the dynamic systems (4), for $i \in \mathcal{N}_1$, and (5), for $i \in \mathcal{N}_2$, to generate the vector $\bar{\mathbf{r}}_i$, which is considered as a first intermediate reference trajectory for the i^{th} agent ($i \in \mathcal{N}$). For the group of agents of second order dynamics, $i \in \mathcal{N}_1$, the additional dynamic systems (6)-(7) are implemented to generate the vectors $\boldsymbol{\zeta}_i$ and $\boldsymbol{\xi}_i$. These auxiliary vectors define a second intermediate reference trajectory $\mathbf{r}_i$ such that: $\mathbf{r}_i = \bar{\mathbf{r}}_i - (\boldsymbol{\zeta}_i + \boldsymbol{\xi}_i)$, for $i \in \mathcal{N}_1$. For the group of agents with first order dynamics, $i \in \mathcal{N}_2$, the dynamic system (7) is implemented to generate $\boldsymbol{\xi}_i$. This vector defines a second intermediate reference trajectory $\mathbf{r}_i$ such that: $\mathbf{r}_i = \bar{\mathbf{r}}_i - \boldsymbol{\xi}_i$, for $i \in \mathcal{N}_2$.

With the above definitions, the input of the dynamic system (7) is designed, without consideration of the input constraints, such that all agents in the team reach an agreement on their second intermediate reference trajectories, *i.e.,* $(\mathbf{r}_i - \mathbf{r}_j) \to 0$ and $\dot{\mathbf{r}}_i \to \dot{\mathbf{p}}_d$, for $i, j \in \mathcal{N}$. To this end, all agents need to transmit their variables $\mathbf{r}_i$, rather than transmitting their real position states. Once this is achieved, the auxiliary variables $\boldsymbol{\xi}_i$, for $i \in \mathcal{N}$, and $\boldsymbol{\zeta}_i$, for $i \in \mathcal{N}_1$, are driven to zero asymptotically, leading, hence, the error between the two intermediate trajectories to converge asymptotically to zero, *i.e.,* $(\mathbf{r}_i - \bar{\mathbf{r}}_i) \to 0$, for $i \in \mathcal{N}$. Finally, the bounded control input for each agent is designed without velocity measurements, as in (10)-(12), to ensure that each agent tracks asymptotically its corresponding first intermediate reference trajectory, $\bar{\mathbf{r}}_i$, achieving hence our control objectives.

To apply the result of Theorem 1, one only needs to design the input of the first-order multi-agent system (9) such that design condition 1 is satisfied under the prescribed communication topology. This input is free from any constraints and is constructed based on available signals. Therefore, Theorem 1 provides a straightforward method extending existent consensus algorithms developed for first-order multi-agent systems, containing identical agents with no input constraints, to solve the consensus problem for the heterogeneous multi-agent system (1), and account for input saturations without velocity measurements.

IV. Application Examples

In this section, we use our main result in the previous section to derive solutions of two different consensus problems for the heterogeneous multi-agent system (1) with input saturations and without velocity measurements.

A. Example I: Consensus with communication delays

We consider the consensus problem of the heterogeneous multi-agent system (1) with communication delays. We assume that the interconnection between agents is represented by the directed graph $\mathcal{G}_n$, and the i-th agent receives information from the j-th agent with a constant delay τ_{ij}. We consider the case where the final velocities of agents are required to converge to zero, *i.e.,* $\dot{\mathbf{p}}_d = 0$. According to Theorem 1, we first design the input of the multi-agent system (9) to satisfy design condition 1 under directed communication topology and in the presence of communication delays. For this purpose, we propose the following function

$$\boldsymbol{\Phi}_{i,\mathcal{G}_n}(\hat{\mathbf{r}}) = -\sum_{j=1}^{n} k_{ij}(\mathbf{r}_i - \mathbf{r}_j(t - \tau_{ij})), \quad (15)$$

with k_{ij} being the $(i,j)^{th}$ entry of the adjacency matrix of the directed graph $\mathcal{G}_n$. We can show that the multi-agent system (9) with (15), and $\dot{\mathbf{p}}_d = 0$, achieves consensus in the presence of arbitrary constant communication delays if the directed communication graph is strongly connected[2]. This

[2]A directed graph is strongly connected if there exists a directed path between any two distinct nodes [21].

can be verified using the Lyapunov-Krasovskii functional

$$V = \frac{1}{2}\sum_{j=1}^{n}\gamma_i\left(\mathbf{r}_i^\top\mathbf{r}_i + \sum_{j=1}^{n}k_{ij}\int_{t-\tau_{ij}}^{t}\mathbf{r}_j^\top(s)\mathbf{r}_j(s)ds\right)$$

having the negative semi-definite time-derivative $\dot{V} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i k_{ij}\|\mathbf{r}_i - \mathbf{r}_j(t-\tau_{ij})\|$, with $\gamma_i > 0$ being the elements of the vector $\gamma := \mathrm{col}(\gamma_1,\ldots,\gamma_n)$ such that $\gamma^\top\mathbf{L} = 0$, where $\mathbf{L} := [l_{ij}] \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of the communication graph defined as: $l_{ii} = \sum_{j=1}^{n}k_{ij}$ and $l_{ij} = -k_{ij}$. Note that for a directed strongly connected graph, such a γ always exists. Following standard signal chasing, we can verify that the right hand side of (15) is bounded and converges asymptotically to zero, $\dot{\mathbf{r}}_i \to 0$, and $(\mathbf{r}_i - \mathbf{r}_j) \to 0$ for all $i,j \in \mathcal{N}$. This satisfies design condition 1.

Consequently, we propose the following input for each agent with second-order dynamics, *i.e.,* $i \in \mathcal{N}_1$,

$$\begin{aligned}
\mathbf{u}_i &= -k_i^d\boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i) - k_i^p\boldsymbol{\chi}(\mathbf{e}_i) \\
&\quad - L_i^\zeta\mathbf{h}(\boldsymbol{\zeta}_i)\left(-L_i^\zeta\boldsymbol{\chi}(\boldsymbol{\zeta}_i) + L_i^\xi\boldsymbol{\chi}(\boldsymbol{\xi}_i)\right), \\
\dot{\boldsymbol{\psi}}_i &= k_i^\psi(\mathbf{e}_i - \boldsymbol{\psi}_i), \\
\ddot{\bar{\mathbf{r}}}_i &= -L_i^\zeta\boldsymbol{\chi}(\boldsymbol{\zeta}_i), \\
\dot{\boldsymbol{\zeta}}_i &= -L_i^\zeta\boldsymbol{\chi}(\boldsymbol{\zeta}_i) + L_i^\xi\boldsymbol{\chi}(\boldsymbol{\xi}_i), \\
\dot{\boldsymbol{\xi}}_i &= -L_i^\xi\boldsymbol{\chi}(\boldsymbol{\xi}_i) + \textstyle\sum_{j=1}^{n}k_{ij}(\mathbf{r}_i - \mathbf{r}_j(t-\tau_{ij})),
\end{aligned} \tag{16}$$

and the following input for all agents with first-order dynamics, *i.e.,* $i \in \mathcal{N}_2$,

$$\begin{aligned}
\mathbf{u}_i &= -L_i^\xi\chi(\boldsymbol{\xi}_i) - k_i^p\boldsymbol{\chi}(\mathbf{e}_i), \\
\dot{\bar{\mathbf{r}}}_i &= -L_i^\xi\boldsymbol{\chi}(\boldsymbol{\xi}_i), \\
\dot{\boldsymbol{\xi}}_i &= -L_i^\xi\boldsymbol{\chi}(\boldsymbol{\xi}_i) + \textstyle\sum_{j=1}^{n}k_{ij}(\mathbf{r}_i - \mathbf{r}_j(t-\tau_{ij}))
\end{aligned} \tag{17}$$

where $\mathbf{e}_i = (\mathbf{p}_i - \bar{\mathbf{r}}_i)$, for $i \in \mathcal{N}$, $\mathbf{r}_i$ is defined in (8), and the control gains are given above. Using the result of Theorem 1, we can conclude that the heterogeneous multi-agent system (1) with the consensus algorithm (16)-(17) achieves consensus, *i.e.,* $(\mathbf{p}_i - \mathbf{p}_j) \to 0$, $\dot{\mathbf{p}}_i \to 0$, for $i,j \in \mathcal{N}$, under a strongly connected directed communication graph with arbitrary constant communication delays. In addition, the control input for each agent is guaranteed to be a priori bounded as in (13) with $\mathbf{v}_{\max}$ and $\mathbf{a}_{\max}$ set to zero.

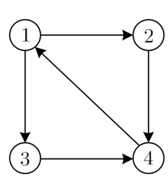

Fig. 1. Interaction graph $\tilde{\mathcal{G}}_4$.

To validate the above result, we implement consensus algorithm (16)-(17) to a group of four agents modeled as in (1), with $m = 1$ and $\mu = 2$, *i.e.,* $\mathcal{N}_1 = \{1,2\}$ and $\mathcal{N}_2 = \{3,4\}$. We assume that all agents are constrained such that $\mathbf{u}_{\max} = 2$, and the information flow between agents is represented by the directed strongly connected graph $\tilde{\mathcal{G}}_4$ given in Fig. 1. The initial conditions of agents are selected as: $\mathbf{P}(0) = (1, 1.5, 2, 3)^\top$ and $\mathbf{V}(0) = (0.1, 0.2)^\top$, with $\mathbf{P}(t) = \mathrm{col}[\mathbf{p}_i(t)]$ for $i \in \mathcal{N} := \{1,2,3,4\}$, and $\mathbf{V}(t) = \mathrm{col}[\dot{\mathbf{p}}_i(t)]$ for $i \in \mathcal{N}_1$. Also, the initial conditions of the auxiliary systems are selected as: $\bar{\mathbf{r}}_i(0) = \mathbf{p}_i(0)$ and $\boldsymbol{\xi}_i(0) = 0$ for $i \in \mathcal{N}$, $\boldsymbol{\zeta}_i(0) = \boldsymbol{\psi}_i(0) = 0$, for $i \in \mathcal{N}_1$. The saturation function in (3) is given as $\sigma(x) = \tanh x$, with $\sigma_b = \sigma_h = 1$, and the control gains are selected as: $k_{ij} = 5$, for $(i,j) \in \mathcal{E}$, $(k_i^p, L_i^\xi) = (0.3, 1)$, for $i \in \mathcal{N}$, and $(k_i^d, k_i^\psi, L_i^\zeta) = (0.45, 1, 0.5)$ for $i \in \mathcal{N}_1$. It is clear that condition (14) is satisfied with $\mathbf{v}_{\max} = 0$ and $\mathbf{a}_{\max} = 0$. The communication delays are considered as: $\tau_{1i} = 0.3$ sec, $\tau_{2i} = 0.2$ sec, $\tau_{3i} = 0.4$ sec, $\tau_{4i} = 0.3$ sec, for $i \in \mathcal{N}$. Fig. 2 shows the systems trajectories in this case, where it can be seen that consensus is achieved despite the communication delays and $|\mathbf{u}_i| \le \mathbf{u}_{\max}$.

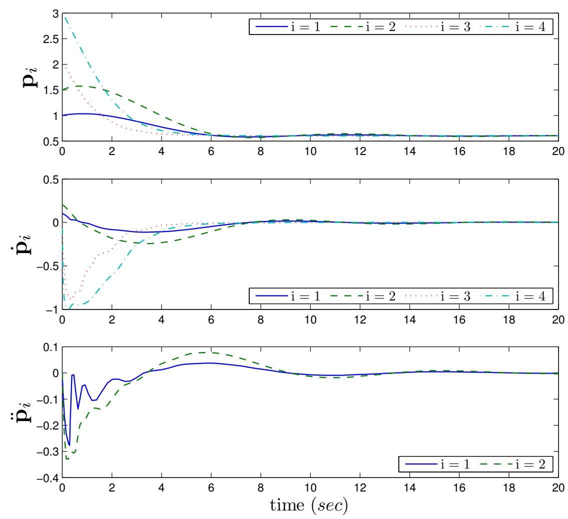

Fig. 2. Simulation results for Example I.

B. Example II: Consensus with a group desired velocity

The control objective in this example is to design a consensus algorithm such that multi-agent system (1) achieves output consensus and each member of the team tracks a common desired velocity, given by $\dot{\mathbf{p}}_d(t)$, and is available to each agent in the team. The desired velocity is assumed to satisfy $\|\dot{\mathbf{p}}_d(t)\|_\infty \le \mathbf{v}_{\max} < \mathbf{u}_{\max}$ and $\|\ddot{\mathbf{p}}_d(t)\|_\infty \le \mathbf{a}_{\max} < \mathbf{u}_{\max}$.

Similar to the previous example, to satisfy Design condition 1, we propose the following function in (9)

$$\boldsymbol{\Phi}_{i,\mathcal{G}_n}(\hat{\mathbf{r}}) = -\sum_{j=1}^{n}k_{ij}(\mathbf{r}_i - \mathbf{r}_j), \tag{18}$$

with k_{ij} being the $(i,j)^{th}$ entry of the adjacency matrix of the directed graph $\mathcal{G}_n$. Let $\tilde{\mathbf{r}}_i = (\mathbf{r}_i - \int_0^t \dot{\mathbf{p}}_d(s)ds)$, and $\dot{\tilde{\mathbf{r}}}_i = (\dot{\mathbf{r}}_i - \dot{\mathbf{p}}_d)$. Therefore, the dynamics of the multi-agent system (9) with (18) can be rewritten as

$$\dot{\tilde{\mathbf{r}}}_i = -\sum_{j=1}^{n}k_{ij}(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j), \quad \text{for } i \in \mathcal{N}. \tag{19}$$

Then, following the same steps as in [4], we can show that $\dot{\bar{\mathbf{r}}}_i$ and $(\mathbf{r}_i - \mathbf{r}_j)$ are globally bounded and $(\mathbf{r}_i - \mathbf{r}_j) \to 0$ and $\dot{\mathbf{r}}_i \to \dot{\mathbf{p}}_d$, if the directed communication graph contains a spanning tree. Therefore, Design condition 1 is satisfied.

As a result, we conclude that the heterogeneous multi-agent system (1) with the consensus algorithm given by (10)-(12), with (4)-(7) and (18), with $\mathbf{r}_i$ given in (8), achieves consensus in the sense of (2), *i.e.*, $(\mathbf{p}_i - \mathbf{p}_j) \to 0$, $\dot{\mathbf{p}}_i \to \dot{\mathbf{p}}_d$, for $i, j \in \mathcal{N}$, under the condition that the directed communication graph has a spanning tree. In addition, the control input for each agent is guaranteed to be a priori bounded as in (13).

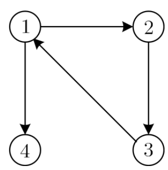

Fig. 3. Interaction graph $\mathcal{G}_4$.

To test the effectiveness of the consensus algorithm in this subsection, we consider the same heterogeneous multi-agent system in Example I, with $\mathbf{u}_{\max} = 2$, under the directed graph $\mathcal{G}_4$ given in Fig.3, which contains a directed spanning tree. The common desired velocity is given as $\dot{\mathbf{p}}_d = 0.5\sin(2t/\pi)$. The initial conditions and the control gains are selected as in Example I. Fig.4 shows the obtained simulation results, where we can see that the heterogeneous multi-agent system reaches consensus and the control input for each agent satisfies $|\mathbf{u}_i| \leq \mathbf{u}_{\max}$.

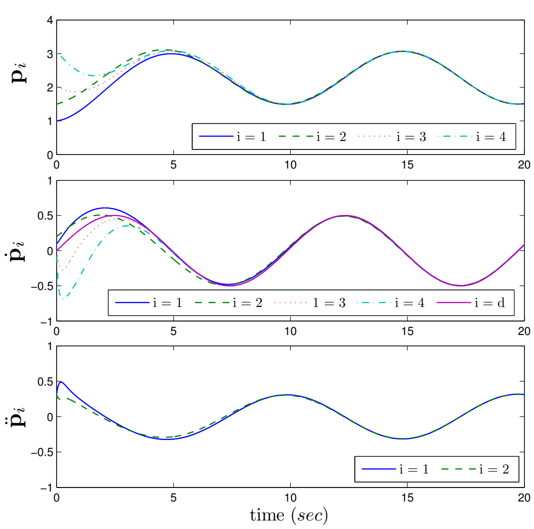

Fig. 4. Simulation results for Example II.

V. Conclusion

We proposed output consensus algorithms for heterogeneous multi-agent systems subject to input saturation constraints. We have shown that first-order consensus algorithms, developed under a certain communication topology and satisfying some conditions, can be extended to the heterogeneous case in the presence of input constraints and without velocity measurements. This has been illustrated in two examples, where solutions to two different consensus problems have been developed. Although we consider heterogeneous multi-agent systems containing agents with single and double integrator dynamics, this work gives insights to a future extension to the case of high order heterogeneous multi-agent systems.

Appendix I
Proof of Theorem 1

First, it can be seen that $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$, for $i \in \mathcal{N}$, is verified from (13) under condition (14).

Consider the error vector $\mathbf{e}_i := (\mathbf{p}_i - \bar{\mathbf{r}}_i)$. By taking the time-derivative of (4) we can notice, in view of (6) and property P3, that $\ddot{\bar{\mathbf{r}}}_i = \ddot{\mathbf{p}}_d - L_i^{\zeta}\mathbf{h}(\boldsymbol{\zeta}_i)\left(-L_i^{\zeta}\boldsymbol{\chi}(\boldsymbol{\zeta}_i) + L_i^{\xi}\boldsymbol{\chi}(\boldsymbol{\xi}_i)\right)$, for $i \in \mathcal{N}_1$. Therefore, using (1), (4)-(5), and (10)-(12), we can write

$$\begin{cases} \ddot{\mathbf{e}}_i = & -k_i^p\boldsymbol{\chi}(\mathbf{e}_i) - k_i^d\boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i), \\ \dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{e}_i - \boldsymbol{\psi}_i), & \end{cases} \tag{A-1}$$

for $i \in \mathcal{N}_1$, and

$$\dot{\mathbf{e}}_i = \; -k_i^p\boldsymbol{\chi}(\mathbf{e}_i) \quad \text{for} \quad i \in \mathcal{N}_2. \tag{A-2}$$

Consider the following positive definite functional

$$\begin{aligned} V = & \frac{1}{2}\sum_{i\in\mathcal{N}_1}\left(\dot{\mathbf{e}}_i^\top\dot{\mathbf{e}}_i + k_i^p\sum_{k=1}^{m}\int_0^{e_i^k}\sigma(s)ds\right) \\ & + \frac{1}{2}\sum_{i\in\mathcal{N}_1}k_i^d\sum_{k=1}^{m}\int_0^{(e_i^k-\psi_i^k)}\sigma(s)ds, \end{aligned} \tag{A-3}$$

with $\mathbf{e}_i = \mathrm{col}[e_i^k]$ and $\boldsymbol{\psi}_i = \mathrm{col}[\psi_i^k]$, for $k \in \{1, ..., m\}$, $i \in \mathcal{N}_1$, and σ is the scalar function defined in (3).

Note that V in (A-3) can be verified to be radially unbounded from the definition of σ. The time-derivative of V evaluated along the dynamics (A-1) can be obtained as

$$\begin{aligned} \dot{V} = & \sum_{i\in\mathcal{N}_1}\dot{\mathbf{e}}_i^\top\left(-k_i^p\boldsymbol{\chi}(\mathbf{e}_i) - k_i^d\boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i)\right) \\ & + \sum_{i\in\mathcal{N}_1}\left(k_i^p\dot{\mathbf{e}}_i^\top\boldsymbol{\chi}(\mathbf{e}_i) + k_i^d(\dot{\mathbf{e}}_i - \dot{\boldsymbol{\psi}}_i)^\top\boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i)\right) \\ = & -\sum_{i\in\mathcal{N}_1}k_i^d k_i^{\psi}(\mathbf{e}_i - \boldsymbol{\psi}_i)^\top\boldsymbol{\chi}(\mathbf{e}_i - \boldsymbol{\psi}_i), \end{aligned} \tag{A-4}$$

which is negative semi-definite, and we conclude that $\dot{\mathbf{e}}_i$, $\mathbf{e}_i$, $\boldsymbol{\psi}_i$, and $\dot{\boldsymbol{\psi}}_i$ are bounded for $i \in \mathcal{N}_1$. This, with property P3, leads us to conclude that $\ddot{V}$ is bounded. Invoking Barbălat Lemma, we conclude that $\dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{e}_i - \boldsymbol{\psi}_i) \to 0$ for $i \in \mathcal{N}_1$. Furthermore, we can verify that $(\ddot{\mathbf{e}}_i - \ddot{\boldsymbol{\psi}}_i)$ is bounded for $i \in \mathcal{N}_1$. Invoking Barbălat Lemma, we conclude that $(\dot{\mathbf{e}}_i - \dot{\boldsymbol{\psi}}_i) \to 0$, for $i \in \mathcal{N}_1$, and hence we know that $\dot{\mathbf{e}}_i \to 0$ for $i \in \mathcal{N}_1$. In addition, we can show from the time-derivative of (A-1) and property P3 that $\dddot{\mathbf{e}}_i$, for $i \in \mathcal{N}_1$, is bounded. Invoking Barbălat Lemma again, we conclude that $\ddot{\mathbf{e}}_i \to 0$, for $i \in \mathcal{N}_1$, which leads us to conclude that $\mathbf{e}_i \to 0$ for all $i \in \mathcal{N}_1$. In addition, it is straightforward to show from (A-2), by exploiting the properties of the function χ, that $\mathbf{e}_i$

(for $i \in \mathcal{N}_2$) is globally bounded and $\mathbf{e}_i \to 0$, $\dot{\mathbf{e}}_i \to 0$ for $i \in \mathcal{N}_2$. As a result, we conclude that $(\mathbf{p}_i - \bar{\mathbf{r}}_i) \to 0$ and $(\dot{\mathbf{p}}_i - \dot{\bar{\mathbf{r}}}_i) \to 0$, for $i \in \mathcal{N}$.

Furthermore, Design condition 1 guarantees that the multi-agent system 9 achieves consensus in the sense that $(\mathbf{r}_i - \mathbf{r}_j) \to 0$, $\dot{\mathbf{r}}_i \to \dot{\mathbf{p}}_d$, for all $i, j \in \mathcal{N}$, and the function $\mathbf{\Phi}_{i,\mathcal{G}_n}(\hat{\mathbf{r}})$ is globally bounded and converges asymptotically to zero for $i \in \mathcal{N}$. Therefore, the dynamics of the auxiliary variable $\boldsymbol{\xi}_i$, given in (7), can be written as

$$\dot{\boldsymbol{\xi}}_i = -L_i^{\xi} \boldsymbol{\chi}(\boldsymbol{\xi}_i) + \boldsymbol{\eta}_i,$$

for $i \in \mathcal{N}$, with $\boldsymbol{\eta}_i := -\mathbf{\Phi}^i_{\mathcal{G}_n}(\hat{\mathbf{r}})$, for $i \in \mathcal{N}$. Note that $\boldsymbol{\eta}_i$ is globally bounded and converges asymptotically to zero. Consequently, $\dot{\boldsymbol{\xi}}_i$, $i \in \mathcal{N}$, is globally bounded. To show that $\boldsymbol{\xi}_i$, $i \in \mathcal{N}$, is globally bounded and converges asymptotically to zero, we consider the Lyapunov-like function candidate

$$\mathcal{W}_i = \frac{1}{2} \boldsymbol{\xi}_i^\top \boldsymbol{\xi}_i, \tag{A-5}$$

for $i \in \mathcal{N}$, with its time derivative obtained as

$$\begin{aligned} \dot{\mathcal{W}}_i = \boldsymbol{\xi}_i^\top \dot{\boldsymbol{\xi}}_i &= -\boldsymbol{\xi}_i^\top \left(L_i^{\xi} \boldsymbol{\chi}(\boldsymbol{\xi}_i) - \boldsymbol{\eta}_i \right) \\ &\leq -\sum_{k=1}^{m} |\xi_i^k| \left(L_i^{\xi} \sigma(|\xi_i^k|) - |\eta_i^k| \right) \end{aligned} \tag{A-6}$$

with $\boldsymbol{\xi}_i = \mathrm{col}[\xi_i^k]$ and $\boldsymbol{\eta}_i = \mathrm{col}[\eta_i^k]$, for $k \in \{1, ..., m\}$, where we have used the property; $x\sigma(x) = |x|\sigma(|x|)$, for any $x \in \mathbb{R}$, to obtain the last inequality. Note that $\dot{\mathcal{W}}_i \leq \|\boldsymbol{\xi}_i\| \|\boldsymbol{\eta}_i\|$, and using the fact that $\|\boldsymbol{\xi}_i\|^2 \leq 2\mathcal{W}_i$, we can write $\dot{\mathcal{W}}_i \leq \|\boldsymbol{\eta}_i\| \sqrt{2\mathcal{W}_i}$, which can be rewritten as $\frac{d\mathcal{W}_i}{\sqrt{\mathcal{W}_i}} \leq \bar{\eta}_i dt$, with $\sqrt{2}\|\boldsymbol{\eta}_i\| \leq \bar{\eta}_i$. Integrating this last inequality over the interval $[t_0, t]$ yields: $2\left(\sqrt{\mathcal{W}_i(t)} - \sqrt{\mathcal{W}_i(t_0)}\right) \leq \bar{\eta}_i(t - t_0)$, which shows that there is no finite time for $\boldsymbol{\xi}_i$, $i \in \mathcal{N}$.

Now, since the function σ is bounded, it is easy to verify that the right hand side of inequality (A-6) is positive when $|\eta_i^k| > \sigma_b L_i^{\xi}$. However, since $\boldsymbol{\eta}_i$ is bounded and converges asymptotically to zero, it is clear that there exists a finite time t_1 such that $|\eta_i^k(t)| \leq \sigma_b L_i^{\xi}$ for all $t \geq t_1$. Note that $\boldsymbol{\xi}_i$ remains bounded on the interval $[0, t_1]$ as there is no finite-escape time. Consequently, for all $t \geq t_1$, one can conclude that the right hand side of (A-6) is negative outside the set $\mathcal{S} = \left\{ \boldsymbol{\xi}_i \mid \sigma(|\xi_i^k|) \leq \frac{|\eta_i^k|}{L_i^d}, \text{ for } k = 1, ..., m \right\}$. Also, we can conclude that $\boldsymbol{\xi}_i$ is bounded outside the set $\mathcal{S}$. Since $\sigma(|.|)$ is a class $\mathcal{K}$ function, $\boldsymbol{\xi}_i$ is ultimately bound to reach the set $\mathcal{S}$ and will be driven to zero as $\boldsymbol{\eta}_i \to 0$. As a result, we conclude that $\boldsymbol{\xi}_i \to 0$ and $\dot{\boldsymbol{\xi}}_i \to 0$ for $i \in \mathcal{N}$.

Consequently, the dynamics of the vector $\boldsymbol{\zeta}_i$ for $i \in \mathcal{N}_1$, given in (6), can be rewritten as

$$\dot{\boldsymbol{\zeta}}_i = -L_i^{\zeta} \boldsymbol{\chi}(\boldsymbol{\zeta}_i) + \tilde{\boldsymbol{\eta}}_i,$$

with $\tilde{\boldsymbol{\eta}}_i := L_i^{\xi} \boldsymbol{\chi}(\boldsymbol{\xi}_i)$, for $i \in \mathcal{N}_1$, is bounded and converges asymptotically to zero. Following the same steps as above, we can conclude that $\dot{\boldsymbol{\zeta}}_i$ and $\boldsymbol{\zeta}_i$ are globally bounded and $\dot{\boldsymbol{\zeta}}_i \to 0$, $\boldsymbol{\zeta}_i \to 0$, for $i \in \mathcal{N}_1$.

Finally, since we have shown that $\dot{\mathbf{r}}_i \to \dot{\mathbf{p}}_d$, $(\mathbf{r}_i - \mathbf{r}_j) \to 0$, for $i, j \in \mathcal{N}$, we can conclude, in view of the definition of $\mathbf{r}_i$ in (8), that $(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_j) \to 0$, $\dot{\bar{\mathbf{r}}}_i \to \dot{\mathbf{p}}_d$, for all $i, j \in \mathcal{N}$. This with the fact that $(\mathbf{p}_i - \bar{\mathbf{r}}_i) \to 0$ and $(\dot{\mathbf{p}}_i - \dot{\bar{\mathbf{r}}}_i) \to 0$, for $i \in \mathcal{N}$, lead us to conclude that $(\mathbf{p}_i - \mathbf{p}_j) \to 0$, $\dot{\mathbf{p}}_i \to \dot{\mathbf{p}}_d$, for all $i, j \in \mathcal{N}$.

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