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# A Unified Adaptive Iterative Learning Control Framework for Uncertain Nonlinear Systems

Abdelhamid Tayebi and Chiang-Ju Chien

***Abstract*—In this note, we propose a unified framework for adaptive iterative learning control design for uncertain nonlinear systems. It is shown that if a Lyapunov based adaptive control law is available for the system under consideration and the Lyapunov function satisfies certain conditions, it is straightforward to extend the adaptive controller to handle repetitive systems operating over a finite time interval. According to the value of a certain parameter $\gamma$, the parametric adaptation law can be a pure time-domain adaptation, a pure iteration-domain adaptation or a combination of both.[1] The advantages and disadvantages of the three possible adaptation types are discussed and some illustrative examples are given.**

***Index Terms*—Adaptive control, iterative learning control, nonlinear systems.**

## I. Introduction

After more that two decades of intensive research, iterative learning control (ILC) is now a well-established control technique that fits well systems that are repetitive in nature. Roughly speaking, this technique aims to generate, in an iterative manner, the adequate control input leading to a "perfect" tracking over a finite time-interval for systems executing repetitive tasks over a finite time-interval (see, for instance, [1]–[4], [13], and [16]). In its early stages, the design of ILC schemes was, primarily, based upon the contraction mapping approach and the use of the time-weighted norm (or $\lambda$-norm) to prove the convergence of the iterative process. This approach basically consists of adjusting the previous control input with an adequate correcting term depending, generally, on the current and/or the previous tracking error profiles. This approach encountered several well-known obstacles, such as the resetting condition, low convergence rates, requirement of the global Lipschitz condition for nonlinear systems, and use of the output time-derivatives for systems with high relative degree. In this framework, the reference trajectory as well as the disturbances are usually assumed to be iteration-invariant [i.e., the reference trajectory (or the disturbance) has to be the same at each iteration].

In the mid-1990s, a new ILC approach, adaptive ILC, based on a Lyapunov-like theory was introduced to overcome some of the limitations of the original approach [5]–[8], [11], [12], [14], [16], [17]. This new design methodology, which inherits the main attributes from its counterpart in standard nonlinear theory, provided powerful tools to handle complex systems that were difficult to handle using the contraction mapping approach. In fact, among the benefits of this approach, one can recall the relaxation of the resetting and Lipschitz conditions, the ability to handle systems with high relative degree, and iteration-varying disturbances and reference trajectories. In this framework, the previous control input is adjusted indirectly through the adjustment of some parameters in the control law. The adjustment of the parameters can be performed along the iteration axis [14], [16], [17], along the time-axis (initializing the parameter estimates with their final values obtained at the preceding iteration) [6], or combining both [7], [8], [11], [12]. In fact, in [7] and [8], only uniform boundedness has been proven for a particular class of uncertain nonlinear systems in lower triangular form. Afterwards, the authors in [11] extended the work of [7] and [8] by proving the stability and the asymptotic convergence of the tracking error and the composite learning error for a class of uncertain nonlinear systems in lower triangular form. In the tutorial paper [11], a combined iteration-domain and time-domain adaptive ILC algorithm has been proposed for a specific class of uncertain nonlinear systems with specific structural properties.

In this note, we provide a unified formulation of adaptive ILC for a quite large class of uncertain nonlinear systems.[2] In fact, we provide a systematic procedure for the design of adaptive ILC schemes for uncertain systems based on the existence of a Lyapunov function for the system under consideration. The proposed parametric adaptation law is quite general in the sense that it depends on a scalar $\gamma$ allowing to select the desired type among the three adaptation types discussed above, namely, a pure time-domain adaptation for $\gamma = 0$, a pure iteration-domain adaptation for $\gamma = 1$, and a combination of both for $\gamma \in (0, 1)$. In this framework, the reference trajectory is allowed to be iteration-varying and the initial tracking error, at each iteration, is set either to zero (resetting condition) or to the tracking error obtained at the end of the previous iteration (alignment condition). The advantages and disadvantages of the three adaptation types are discussed, and some examples illustrating the design procedure are provided. A preliminary version of this note has been presented in [15].

## II. Adaptive ILC Design

Let us consider the following nonlinear system:

$$\dot{x}_k(t) = f(x_k(t), u_k(t), \theta, t) \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector (denoting generally the tracking error), $u_k \in \mathbb{R}^m$ is the control vector, $\theta \in \mathbb{R}^p$ is an unknown constant vector, $t \in [0, T]$ is the time, and $k \in \mathbb{Z}_+$ in the iteration (or trial) index. The nonlinear function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times [0, T] \to \mathbb{R}^n$ is

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[1] A pure iteration-domain adaptation is described by a difference equation, a pure time-domain adaptation is described by a differential equation, and a combination of both is described by a differential-difference equation.

[2] No structural assumptions, such as those used in [7], [8], [11], and [12], are made. The proposed proof does not involve directly the structural properties of the system under consideration.

such that $f(x_k(t), u_k(t), \theta, t)$ is bounded over [0,$T$] as long as $x_k(t)$ and $u_k(t)$ are bounded over [0,$T$]. In general, (1) represents the error dynamics, and the explicit appearance of the time in (1) is often due to the time-varying reference trajectory.

Suppose that one can design a well-defined[3] dynamic control law of the form

$$\begin{aligned} u_k(t) &= g(x_k(t), \hat{\theta}_k(t), t) \\ \dot{\hat{\theta}}_k(t) &= h(x_k(t), t) \end{aligned} \tag{2}$$

such that there exist a positive definite function

$$\Phi(x_k, \tilde{\theta}_k) = V(x_k) + W(\tilde{\theta}_k) \tag{3}$$

with $V$ and $W$ being two differentiable positive definite functions, satisfying

$$\dot{\Phi} = L_f V(x_k) + L_h W(\tilde{\theta}_k) \leq -\Upsilon(x_k) \tag{4}$$

where $\Upsilon(x_k)$ is a positive definite function, $\tilde{\theta}_k = \hat{\theta}_k - \theta$, $L_f V \equiv (\partial V/\partial x_k) f$, and $L_h W \equiv (\partial W/\partial \tilde{\theta}_k) h$.

We assume that $h(x_k(t), t)$ is bounded over [0,$T$] as long as $x_k(t)$ is bounded over [0,$T$]. We also assume that $W(\tilde{\theta}_k)$ satisfies the following properties:

P1)

$$\left\| \frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} \right\|^2 \leq \frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} \tilde{\theta}_k \tag{5}$$

P2)

$$W(\tilde{\theta}_k) - W(\tilde{\theta}_{k-1}) \leq -\Omega(\bar{\theta}_k) + \frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} \bar{\theta}_k \tag{6}$$

where $\Omega(\bar{\theta}_k)$ is a positive semidefinite function and $\bar{\theta}_k(t) = \hat{\theta}_k(t) - \hat{\theta}_{k-1}(t)$.

Note that properties P1) and P2) are purely technical, needed for the proof of our theorem. They are unnecessary in certain situations, as will be discussed later in our remarks. For instance, these properties are satisfied if we consider $W(\tilde{\theta}_k) = (1/2)\tilde{\theta}_k^T \Gamma^{-1} \tilde{\theta}_k$, with $\Gamma$ being a symmetric positive definite matrix. In this particular case, $(\partial W(\tilde{\theta}_k)/\partial \tilde{\theta}_k) = \tilde{\theta}_k^T \Gamma^{-1}$ and $W(\tilde{\theta}_k(t)) - W(\tilde{\theta}_{k-1}(t)) = -(1/2)\bar{\theta}_k^T \Gamma^{-1} \bar{\theta}_k + \bar{\theta}_k^T \Gamma^{-1} \tilde{\theta}_k$.

Throughout this note, we will use the $\mathcal{L}_{pe}$ norm defined as follows:

$$\|x(t)\|_{pe} \triangleq \begin{cases} \left( \int_0^t \|x(\tau)\|^p d\tau \right)^{1/p}, & \text{if } p \in [1, \infty) \\ \sup_{0 \leq \tau \leq t} \|x(\tau)\|, & \text{if } p = \infty \end{cases}$$

where $\|x\|$ denotes any consistent norm of $x$ and $t$ belongs to the finite interval [0,$T$]. We say that $x \in \mathcal{L}_{pe}$ when $\|x\|_{pe}$ exists (i.e., when $\|x\|_{pe}$ is finite).

Now, one can state our result in the following theorem.

*Theorem 1:* Consider system (1) under the following adaptive ILC scheme:

$$\begin{aligned} u_k(t) &= g(x_k(t), \hat{\theta}_k(t), t) \\ (1-\gamma)\dot{\hat{\theta}}_k(t) &= -\gamma \hat{\theta}_k(t) + \gamma \hat{\theta}_{k-1}(t) + h(x_k(t), t) \end{aligned} \tag{7}$$

with $\gamma \in [0, 1]$, $\hat{\theta}_{-1}(t) = 0$. For $\gamma \in [0, 1)$, we set $\hat{\theta}_k(0) = \hat{\theta}_{k-1}(T)$. Assume that $x_k(0) = 0$ or $x_k(0) = x_{k-1}(T)$, $\forall k \in \mathbb{Z}_+$. Then we have the following.

i) For $\gamma \in [0, 1)$, $x_k(t), \tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{\infty e}$, for all $k \in \mathbb{Z}_+$ and for all $t \in [0, T]$, and $\lim_{k \to \infty} x_k(t) = 0$, $\forall t \in [0, T]$.

ii) For $\gamma = 1$, $x_k(t) \in \mathcal{L}_{\infty e}$, $\tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{2e}$ for all $k \in \mathbb{Z}_+$ and for all $t \in [0, T]$, and $\lim_{k \to \infty} \int_0^T \Upsilon(x_k(\tau))\, d\tau = 0$. Moreover, with $x_k(0) = 0$, we have $\lim_{k \to \infty} x_k(t) = 0$, $\forall t \in [0, T]$.

*Proof:* First, we will prove $(i)$, i.e., for $\gamma \in [0, 1)$. Let us consider the following positive definite function:

$$\Psi(x_k, \tilde{\theta}_k) = V(x_k) + (1-\gamma) W(\tilde{\theta}_k). \tag{8}$$

In the sequel, we will use $\Psi_k(t)$ to denote $\Psi(x_k(t), \tilde{\theta}_k(t))$, $V_k(t)$ to denote $V(x_k(t))$, and $W_k(t)$ to denote $W(\tilde{\theta}_k(t))$. The time derivative of (8), in view of (1)–(5), is given by

$$\begin{aligned} \dot{\Psi}_k &= L_f V_k + (1-\gamma) \frac{\partial W_k}{\partial \tilde{\theta}_k} \dot{\hat{\theta}}_k \\ &= L_f V_k + L_h W_k + \frac{\partial W_k}{\partial \tilde{\theta}_k} (-\gamma \hat{\theta}_k + \gamma \hat{\theta}_{k-1}) \\ &\leq -\Upsilon(x_k) + \frac{\partial W_k}{\partial \tilde{\theta}_k} (-\gamma \hat{\theta}_k + \gamma \hat{\theta}_{k-1}) \\ &\leq -\gamma \frac{\partial W_k}{\partial \tilde{\theta}_k} (\hat{\theta}_k - \hat{\theta}_{k-1}) = -\gamma \frac{\partial W_k}{\partial \tilde{\theta}_k} (\tilde{\theta}_k - \tilde{\theta}_{k-1}). \end{aligned}$$

Using property P1) and a simplified version of Young's inequality, i.e., $(\partial W_k/\partial \tilde{\theta}_k) \tilde{\theta}_{k-1} \leq \|(\partial W_k/\partial \tilde{\theta}_k)\|^2 + (1/4)\|\tilde{\theta}_{k-1}\|^2$, we have

$$\begin{aligned} \dot{\Psi}_k &\leq -\gamma \frac{\partial W_k}{\partial \tilde{\theta}_k} \tilde{\theta}_k + \gamma \left\| \frac{\partial W_k}{\partial \tilde{\theta}_k} \right\|^2 + \frac{\gamma}{4} \|\tilde{\theta}_{k-1}\|^2 \\ &\leq \frac{\gamma}{4} \|\tilde{\theta}_{k-1}\|^2. \end{aligned} \tag{9}$$

Since $\hat{\theta}_{-1}(t) = 0$ and $\hat{\theta}_0(0) = \hat{\theta}_{-1}(T)$, it is clear that $\Psi_0(t)$ and hence $x_0(t)$ and $\tilde{\theta}_0(t)$ are bounded for all $t \in [0, T]$.

Now, let us use the following positive definite functional:

$$\bar{\Psi}(x_k, \tilde{\theta}_k, t) = \Psi(x_k, \tilde{\theta}_k) + \gamma \int_0^t W(\tilde{\theta}_k(\tau))\, d\tau \tag{10}$$

whose difference can be evaluated, in view of (1)–(6), as follows:

$$\begin{aligned} \Delta \bar{\Psi}_k(t) &= \Psi_k(t) - \Psi_{k-1}(t) + \gamma \int_0^t (W_k(\tau) - W_{k-1}(\tau))\, d\tau \\ &= V_k(t) - V_{k-1}(t) + (1-\gamma)(W_k(t) - W_{k-1}(t)) \\ &\quad + \gamma \int_0^t (W_k(\tau) - W_{k-1}(\tau))\, d\tau \\ &= -V_{k-1}(t) - (1-\gamma) W_{k-1}(t) + V_k(0) \\ &\quad + (1-\gamma) W_k(0) + \gamma \int_0^t (W_k(\tau) - W_{k-1}(\tau))\, d\tau \\ &\quad + \int_0^t \left( L_f V_k(\tau) + (1-\gamma) \frac{\partial W_k}{\partial \tilde{\theta}_k} \dot{\hat{\theta}}_k(\tau) \right) d\tau \\ &\leq -\int_0^t \Upsilon(x_k(\tau))\, d\tau - \gamma \int_0^t \Omega(\bar{\theta}_k)\, d\tau + V_k(0) - V_{k-1}(t) \\ &\quad + (1-\gamma)(W_k(0) - W_{k-1}(t)). \end{aligned} \tag{11}$$

Now, using the fact that $V_k(0) = 0$ (or $V_k(0) = V_{k-1}(T)$) and $W_k(0) = W_{k-1}(T)$, we have

$$\Delta \bar{\Psi}_k(T) \leq -\int_0^T \Upsilon(x_k(\tau))\, d\tau - \gamma \int_0^T \Omega(\bar{\theta}_k(\tau))\, d\tau \leq 0. \tag{12}$$

Hence, $\bar{\Psi}_k(T)$ is bounded for all $k \in \mathbb{Z}_+$ since $\bar{\Psi}_0(T)$ is bounded due to the boundedness of $\Psi_0(t)$ over $[0, T]$. This implies that $\Psi_k(T)$ and $\int_0^T W(\tilde{\theta}_k(\tau))\, d\tau$ are bounded for all $k \in \mathbb{Z}_+$, which in turn implies that $\int_0^T \|\tilde{\theta}_k\|^2 d\tau$ is bounded for all $k \in \mathbb{Z}_+$ since $W$ is a positive

[3] $g(x_k(t), \hat{\theta}_k(t), t)$ and $h(x_k(t), t)$ are bounded as long as $x_k(t)$, $\hat{\theta}_k(t)$ and $t$ are bounded.

definite function. Now, from (9), in the case where $V_k(0) = V_{k-1}(T)$, one has

$$\Psi_k(t) \le \Psi_k(0) + \int_0^t \frac{\gamma}{4}\|\tilde{\theta}_{k-1}\|^2 d\tau \le \Psi_{k-1}(T) + \int_0^T \frac{\gamma}{4}\|\tilde{\theta}_{k-1}\|^2 d\tau$$

and in the case where $V_k(0) = 0$, one has

$$\Psi_k(t) \le (1-\gamma)W(\tilde{\theta}_{k-1}(T)) + \int_0^T \frac{\gamma}{4}\|\tilde{\theta}_{k-1}\|^2 d\tau$$

which implies that $\Psi_k(t)$ is bounded for all $k \in \mathbb{Z}_+$ and all $t \in [0,T]$, and hence $x_k(t), \tilde{\theta}_k(t), u_k(t)$ are bounded for all $k \in \mathbb{Z}_+$ and for all $t \in [0,T]$. Now, from (12), it is easily seen that

$$\begin{aligned}\bar{\Psi}_k(T) &= \bar{\Psi}_k(0) + \sum_{j=1}^{k} \Delta\bar{\Psi}_j(T) \\ &\le \bar{\Psi}_k(0) - \sum_{j=1}^{k}\int_0^T \Upsilon(x_j(\tau))\,d\tau - \gamma\sum_{j=1}^{k}\int_0^T \Omega(\bar{\theta}_j(\tau))\,d\tau.\end{aligned} \tag{13}$$

Hence

$$\sum_{j=1}^{k}\int_0^T \Upsilon(x_j(\tau))\,d\tau + \gamma\sum_{j=1}^{k}\int_0^T \Omega(\bar{\theta}_j(\tau))\,d\tau \le \bar{\Psi}_k(0) - \bar{\Psi}_k(T). \tag{14}$$

Since $\Psi_k(t)$ is bounded for all $k \in \mathbb{Z}_+$ and for all $t \in [0,T]$, it is clear that $\bar{\Psi}_k(t)$ is bounded for all $k \in \mathbb{Z}_+$ and for all $t \in [0,T]$. Therefore, from (14), one can conclude that $\lim_{k\to\infty}\int_0^T \Upsilon(x_k(\tau))\,d\tau = 0$ and, in the case where $\gamma \ne 0$, $\lim_{k\to\infty}\int_0^T \Omega(\bar{\theta}_k(\tau))d\tau = 0$. Since $x_k(t), \tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{\infty e}$, one has $\dot{x}_k(t) \in \mathcal{L}_{\infty e}$. Consequently, one can conclude that $\lim_{k\to\infty}\Upsilon(x_k(t)) = 0$ for all $t \in [0,T]$ and hence $\lim_{k\to\infty} x_k(t) = 0$ for all $t \in [0,T]$.

Now, let us prove ii), i.e., for $\gamma = 1$. Consider the following positive definite functional:

$$\bar{\Psi}(x_k, \tilde{\theta}_k, t) = V_k(t) + \int_0^t W(\tilde{\theta}_k(\tau))\,d\tau \tag{15}$$

whose time derivative, in view of (1)–(5) and (6), is given by

$$\begin{aligned}\dot{\bar{\Psi}}_k(t) &= \dot{V}_k(t) + W_k(t) \\ &= L_f V_k + W_k(t) - W_{k-1}(t) + W_{k-1}(t) \\ &\le L_f V_k - \Omega(\bar{\theta}_k) + \frac{\partial W(\tilde{\theta}_k)}{\partial \tilde{\theta}_k}\bar{\theta}_k + W_{k-1}(t) \\ &= L_f V_k + L_h W_k - \Omega(\bar{\theta}_k) + W_{k-1}(t) \\ &\le -\Upsilon(x_k) - \Omega(\bar{\theta}_k) + W_{k-1}(t) \le W_{k-1}(t).\end{aligned} \tag{16}$$

Since $\hat{\theta}_{-1}(t) = 0$, it is clear that $\hat{\theta}_0(t) = h(x_0(t), t)$. Since $x_0(0)$ is bounded, it is clear that $\hat{\theta}_0(0)$ is bounded. Therefore, from (16), it is clear that $\bar{\Psi}_0(t)$ is bounded for all $t \in [0,T]$. The difference of $\bar{\Psi}_k(t)$ can be evaluated, in view of (1)–(4) and (6), as follows:

$$\begin{aligned}\Delta\bar{\Psi}_k(t) &= V_k(t) - V_{k-1}(t) + \int_0^t (W_k(\tau) - W_{k-1}(\tau))\,d\tau \\ &= -V_{k-1}(t) + V_k(0) + \int_0^t L_f V_k(\tau)\,d\tau \\ &\quad + \int_0^t (W_k(\tau) - W_{k-1}(\tau))\,d\tau \\ &\le -\int_0^t \Upsilon(x_k(\tau))\,d\tau - \int_0^t \Omega(\bar{\theta}_k)\,d\tau - V_{k-1}(t) + V_k(0)\end{aligned} \tag{17}$$

Now, using the fact that $V_k(0) = 0$ (or $V_k(0) = V_{k-1}(T)$), we have

$$\Delta\bar{\Psi}_k(T) \le -\int_0^T \Upsilon(x_k(\tau))\,d\tau - \int_0^T \Omega(\bar{\theta}_k(\tau))\,d\tau \le 0 \tag{18}$$

which implies that $\bar{\Psi}_k(T)$ is bounded for all $k \in \mathbb{Z}_+$ since $\bar{\Psi}_0(T)$ is bounded.

Let $\varpi_k(t) = \int_0^t W(\tilde{\theta}_k(\tau))\,d\tau$. It is clear that $\varpi_k(t) \le \varpi_k(T) \le \varpi < \infty$ for all $t \in [0,T]$. Therefore

$$\bar{\Psi}_k(t) = V_k(t) + \varpi_k(t) \le V_k(t) + \varpi. \tag{19}$$

Thus

$$\bar{\Psi}_{k-1}(t) \le V_{k-1}(t) + \varpi. \tag{20}$$

On the other hand, one has

$$\begin{aligned}\Delta\bar{\Psi}_k(t) &= \bar{\Psi}_k(t) - \bar{\Psi}_{k-1}(t) \\ &\le -\int_0^t \Upsilon(x_k(\tau))\,d\tau - \int_0^t \Omega(\bar{\theta}_k)\,d\tau - V_{k-1}(t) + V_k(0) \\ &\le V_k(0) - V_{k-1}(t).\end{aligned} \tag{21}$$

From (20) and (21), one can conclude that

$$\bar{\Psi}_k(t) \le V_k(0) + \varpi. \tag{22}$$

Case 1) ($V_k(0) = V_{k-1}(T)$): Since $\bar{\Psi}_k(T)$ is bounded $\forall k \in \mathbb{Z}_+$, it is clear that $V_k(T)$ is bounded $\forall k \in \mathbb{Z}_+$. Hence, from (22), one can conclude that $\bar{\Psi}_k(t)$ is bounded $\forall k \in \mathbb{Z}_+$, $\forall t \in [0,T]$.

Case 2) ($V_k(0) = 0$): It is clear that $\bar{\Psi}_k(t)$ is bounded $\forall k \in \mathbb{Z}_+$, $\forall t \in [0,T]$.

Consequently, $x_k(t) \in \mathcal{L}_{\infty e}$, $\tilde{\theta}_k(t), u_k(t) \in \mathcal{L}_{2e}$ for all $k \in \mathbb{Z}_+$ and for all $t \in [0,T]$.

We have also

$$\sum_{j=1}^{k}\int_0^T \Upsilon(x_j(\tau))\,d\tau + \sum_{j=1}^{k}\int_0^T \Omega(\bar{\theta}_j(\tau))\,d\tau \le \bar{\Psi}_k(0) - \bar{\Psi}_k(T) \tag{23}$$

which implies that $\lim_{k\to\infty}\int_0^T \Upsilon(x_k(\tau))d\tau = 0$ and $\lim_{k\to\infty}\int_0^T \Omega(\bar{\theta}_k(\tau))d\tau = 0$.

Now, let us show that $\lim_{k\to\infty} x_k(t) = 0$, $\forall t \in [0,T]$, in the case where $x_k(0) = 0$, $\forall k \in \mathbb{Z}_+$. In fact, in this case, (17) leads to

$$\Delta\bar{\Psi}_k(t) \le -V_{k-1}(t) - \int_0^t \Upsilon(x_k(\tau))\,d\tau - \int_0^t \Omega(\bar{\theta}_k(\tau))\,d\tau \le 0 \tag{24}$$

which leads to

$$\begin{aligned}&\sum_{j=1}^{k} V_{j-1}(t) + \sum_{j=1}^{k}\int_0^t \Upsilon(x_j(\tau))\,d\tau \\ &\qquad + \sum_{j=1}^{k}\int_0^t \Omega(\bar{\theta}_j(\tau))\,d\tau \le \bar{\Psi}_k(0) - \bar{\Psi}_k(t).\end{aligned} \tag{25}$$

Since $\bar{\Psi}_k(t)$ is bounded $\forall k \in \mathbb{Z}_+$, $\forall t \in [0,T]$, it is clear that $\lim_{k\to\infty} V_k(t) = 0$, which implies that $\lim_{k\to\infty} x_k(t) = 0$, $\forall t \in [0,T]$.

Now, we would like to provide the following remarks.

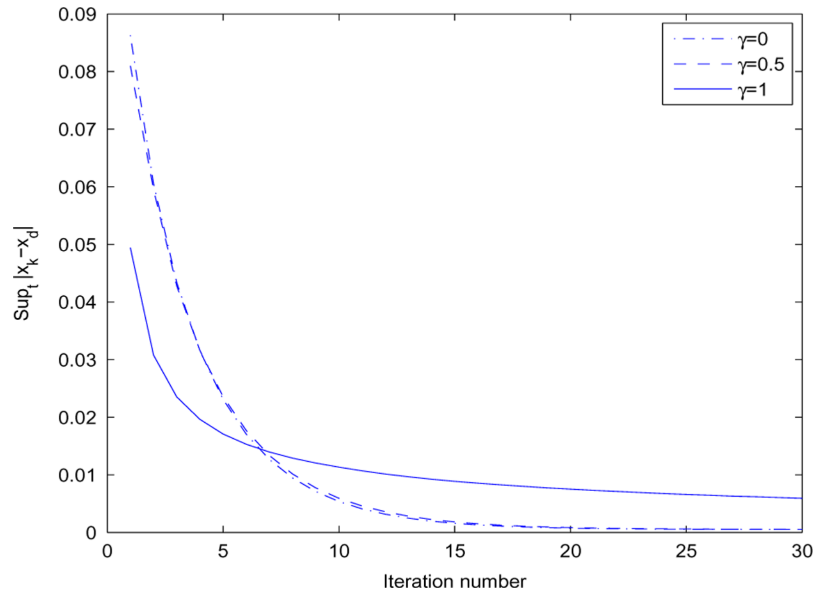


Fig. 1. Example 1: Sup-norm of the tracking error versus the iteration number for $\gamma = 0, 0.5, 1$.

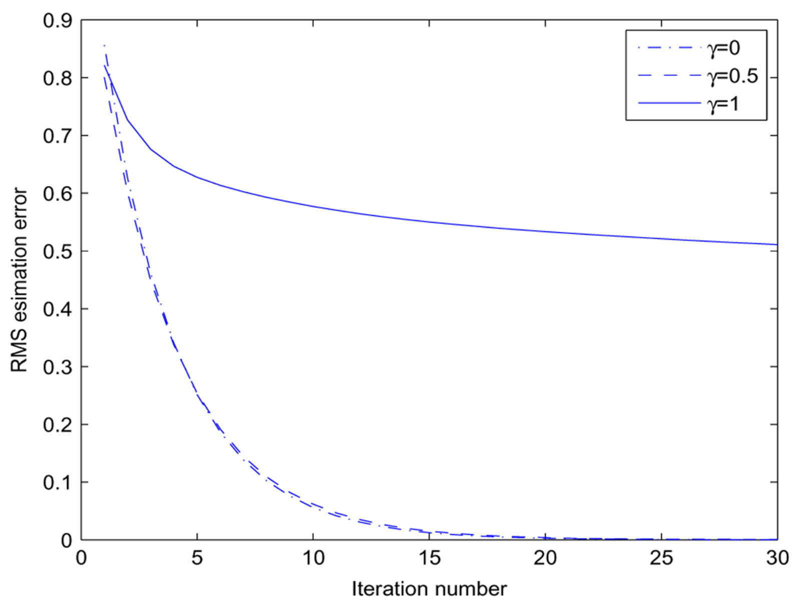


Fig. 2. Example 1: RMS-norm of the estimation error $(\tilde{\theta}_k)$ versus the iteration number for $\gamma = 0, 0.5, 1$.

*Remark 1:* Note that the parametric adaptation law given in (7) is a hybrid time-domain and iteration-domain adaptation mechanism for $\gamma \in (0,1)$. In the case where $\gamma = 0$, the adaptation law becomes a pure time-domain adaptation [6], while for $\gamma = 1$, it becomes a pure iteration-domain adaptation [16]. With $\gamma \in [0,1)$, we guarantee the boundedness of the infinity norm of the tracking error and the control input as well as the convergence to zero of the infinity norm of the tracking error. With $\gamma = 1$, we guarantee the boundedness of the infinity norm of the tracking error, the boundedness of the $\mathcal{L}_2$-norm of the control input, and the convergence to zero of the $\mathcal{L}_2$-norm of the tracking error if the alignment condition is satisfied and the convergence to zero of the tracking error if the resetting condition is satisfied. It is worth noting that, in the case where an upper bound of the parameter $\theta$ is known, i.e., $\|\theta\| < \theta_m$, one can guarantee the boundedness of the infinity-norm of the control input, with $\gamma = 1$, by using a projection mechanism in the parametric adaptation law.

*Remark 2:* With $\gamma = 1$, property P1) is not required to derive the result in Theorem 1, and the unknown vector $\theta$ in (1) can be time-varying. For $\gamma = 0$, both properties P1) and P2) are not required to derive the result in Theorem 1.

*Remark 3:* It is worth noting that with $\gamma = 0$, we will need to save only $\hat{\theta}_k(T)$ in the memory instead of saving $\hat{\theta}_k(t), t \in [0,T]$. This will considerably contribute to memory space saving in real-time applications.

*Remark 4:* In the case of $\gamma \in [0,1)$, it is possible to guarantee the convergence of the parametric estimation error to zero when $k$ tends to infinity, for all $t \in [0,T]$, if the control law (2) guarantees some form of persistency of excitation. In fact, according to the proof of item $(i)$ of our theorem, we have $\lim_{k\to\infty} \int_0^T \Omega(\bar{\theta}_k(\tau))d\tau = 0$ for $\gamma \in (0,1)$. Since all the signals are bounded, it is possible to show that $\dot{\hat{\theta}}_k(t)$ is bounded, and hence $\lim_{k\to\infty} \bar{\theta}_k(t) = 0$ for all $t \in [0,T]$ as long as $\Omega$ is positive definite. Therefore, when $k$ tends to infinity, the adaptive ILC

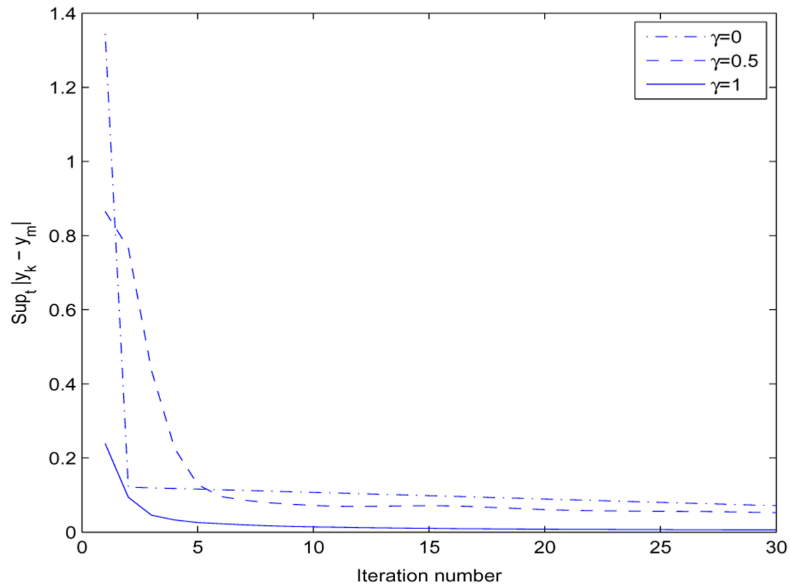


Fig. 3. Example 2: Sup-norm of the tracking error versus the iteration number for $\gamma = 0, 0.5, 1$.

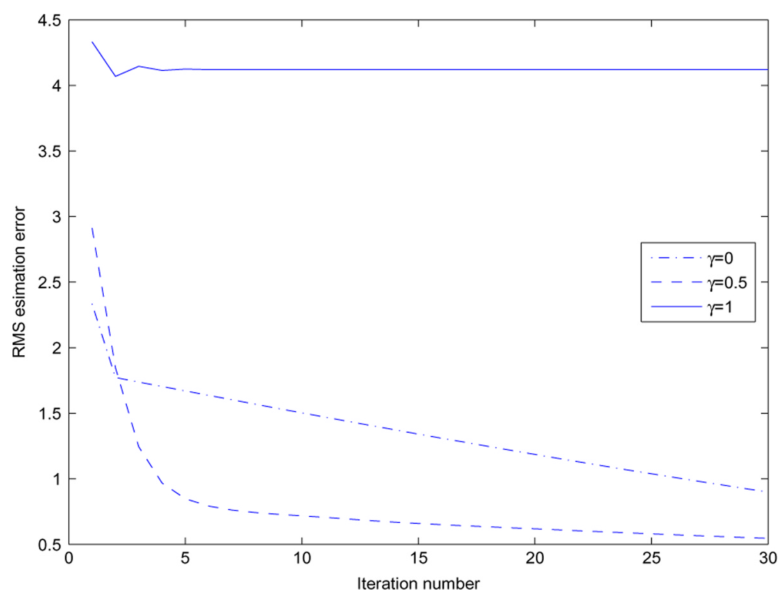


Fig. 4. Example 2: RMS-norm of the estimation error (i.e., $(1/2)(\mathrm{rms}(\tilde{\theta}_{1,k}) + \mathrm{rms}(\tilde{\theta}_{2,k}))$ versus the iteration number for $\gamma = 0, 0.5, 1$.

(7) reduces, in the case of $\gamma \in [0, 1)$, to a pure time-domain integral-type adaptive control with $\hat{\theta}_k(0) = \hat{\theta}_{k-1}(T)$. On the other hand, in the case where $\gamma = 1$, it is not always possible to achieve the convergence of the parametric estimation error to zero when $k$ tends to infinity, for all $t \in [0, T]$. In fact, one can easily see that $\hat{\theta}_k(0) = 0$ for all $k$ if $h(x_k(0), 0) = 0$ for all $k$.

*Remark 5:* It is well known in adaptive control that parameter drift is a major issue associated to noise and disturbances. This problem may occur in practical applications with $\gamma = 0$. Several techniques have been proposed in the literature to deal with this problem (e.g., dead zone, projection, leakage, or $\sigma$ modification). Our parametric adaptation law in (7), in the case where $\gamma \in (0, 1)$, contains a leakage term ($\sigma$-modification) $(\gamma/(\gamma-1))\hat{\theta}_k$ that helps eliminate the parameter drift in practical applications (see, for instance, [9]).

*Remark 6:* Using a pure iteration-domain adaptation (i.e., $\gamma = 1$) will avoid the use of the integral to calculate $\hat{\theta}_k$. This is very helpful in real-time applications, since the use of an approximative numerical integration is avoided.

*Remark 7:* In the case where $\gamma \in [0, 1)$, $h$ is allowed to depend on $\hat{\theta}_k$, i.e., $h(x_k(t), \hat{\theta}_k(t), t)$. In the case where $\gamma = 1$, one can also allow $h$ to depend on $\hat{\theta}_k$ if we assume that the following adaptation law:

$$\hat{\theta}_k(t) = \hat{\theta}_{k-1}(t) + h(x_k(t), \hat{\theta}_k(t), t) \tag{26}$$

has a unique solution $\hat{\theta}_k(t)$, which is bounded over [0,$T$] if $\hat{\theta}_{k-1}(t)$ and $x_k(t)$ are bounded over [0,$T$].

*Remark 8:* It is not straightforward to conclude about the convergence rates, achieved with the ILC scheme (7), in terms of $\gamma$. In fact, it depends on the system under consideration as illustrated in our simulation results, where we can clearly see that the best convergence rates, for example 1, are obtained with $\gamma \in [0, 1)$ and the best convergence rate, for example 2, is obtained with $\gamma = 1$.

## III. Illustrative Examples

*Example 1:* Consider the following system:

$$\dot{x}_k = \theta x_k^2 + u_k \tag{27}$$

where $x_k \in \mathbb{R}$, $u_k \in \mathbb{R}$, and $\theta \in \mathbb{R}$ is an unknown constant. Assume that the reference trajectory is given by $x_d(t)$. Assume that $x_d(t)$ and $\dot{x}_d(t)$ are bounded over $[0, T]$. The error dynamics is then given by

$$\dot{\tilde{x}}_k = \theta(\tilde{x}_k + x_d(t))^2 + u_k - \dot{x}_d(t) \tag{28}$$

where $\tilde{x}_k = x_k - x_d$. Under the following control law:

$$\begin{aligned} u_k(t) &= -\hat{\theta}_k(t)(\tilde{x}_k + x_d(t))^2 - k\tilde{x}_k + \dot{x}_d(t) \\ \dot{\hat{\theta}}_k(t) &= \beta\tilde{x}_k(\tilde{x}_k + x_d(t))^2 \end{aligned} \tag{29}$$

with $k > 0$, the following positive definite function:

$$\Phi(\tilde{x}_k, \tilde{\theta}_k) = \frac{1}{2}\tilde{x}_k^2 + \frac{1}{2\beta}\tilde{\theta}_k^2 \tag{30}$$

satisfies

$$\dot{\Phi} = L_f V(\tilde{x}_k) + L_h W(\tilde{\theta}_k) \leq -k\tilde{x}_k^2. \tag{31}$$

Hence, the adaptive ILC leading to the results in Theorem 1 is given by

$$\begin{aligned} u_k(t) &= -\hat{\theta}_k(t)(\tilde{x}_k + x_d(t))^2 - k\tilde{x}_k + \dot{x}_d(t) \\ (1-\gamma)\dot{\hat{\theta}}_k(t) &= -\gamma\hat{\theta}_k(t) + \gamma\hat{\theta}_{k-1}(t) + \beta\tilde{x}_k(\tilde{x}_k + x_d(t))^2 \end{aligned} \tag{32}$$

with $\beta > 0$, $\gamma \in [0, 1]$, and $\hat{\theta}_{-1}(t) = 0$. For $\gamma \in [0, 1)$, we set $\hat{\theta}_k(0) = \hat{\theta}_{k-1}(T)$. Note that $\theta$ is allowed to be time-varying in the case $\gamma = 1$.

*Example 2:* Consider the following system:

$$\dot{y}_k = a_p y_k + k_p u_k \tag{33}$$

where $y_k \in \mathbb{R}$, $u_k \in \mathbb{R}$. The parameters $a_p$ and $k_p \neq 0$ are unknown. We assume that the sign of $k_p$ is known. The objective is to make $y_k$ track the output of the following stable reference model:

$$\dot{y}_m = a_m y_m + k_m r \tag{34}$$

where $r(t)$ is a bounded reference input.

Consider the following direct model reference adaptive controller:

$$\begin{aligned} u_k(t) &= \hat{\theta}_{1,k}(t)r(t) + \hat{\theta}_{2,k}y_k(t) \\ \dot{\hat{\theta}}_{1,k}(t) &= -\mathrm{sgn}(k_p)\alpha e_k(t)r(t) \\ \dot{\hat{\theta}}_{2,k}(t) &= -\mathrm{sgn}(k_p)\alpha e_k(t)y_k(t) \end{aligned} \tag{35}$$

where $\alpha > 0$ and $e_k = y_k - y_m$. Under this controller, the following Lyapunov function:

$$\Phi = \frac{1}{2|k_p|}e_k^2 + \frac{1}{2\alpha}\left(\tilde{\theta}_{1,k}^2 + \tilde{\theta}_{2,k}^2\right) \tag{36}$$

where $\tilde{\theta}_{1,k} = \hat{\theta}_{1,k} - \theta_1^\star$, $\tilde{\theta}_{2,k} = \hat{\theta}_{2,k} - \theta_2^\star$, $\theta_1^\star = (k_m/k_p)$ and $\theta_2^\star = ((a_m - a_p)/k_p)$ leads to

$$\dot{\Phi} = \frac{a_m}{|k_p|}e_k^2 \leq 0. \tag{37}$$

Hence, the adaptive ILC leading to the results in Theorem 1 is given by

$$\begin{aligned} u_k(t) &= \hat{\theta}_{1,k}(t)r(t) + \hat{\theta}_{2,k}y_k(t) \\ (1-\gamma)\dot{\hat{\theta}}_{1,k} &= -\gamma\hat{\theta}_{1,k} + \gamma\hat{\theta}_{1,k-1} - \mathrm{sgn}(k_p)\alpha e_k(t)r(t) \\ (1-\gamma)\dot{\hat{\theta}}_{2,k} &= -\gamma\hat{\theta}_{2,k} + \gamma\hat{\theta}_{2,k-1} - \mathrm{sgn}(k_p)\alpha e_k(t)y_k(t). \end{aligned} \tag{38}$$

*Example 3:* Consider the following uncertain nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1(x_1)^T\theta \\ \dot{x}_2 &= u + \phi_2(x_1, x_2)^T\theta \\ y &= x_1 \end{aligned} \tag{39}$$

where $\theta$ is an unknown constant vector of dimension $p$ and $\phi_1$ and $\phi_2$ are known vectors of smooth functions of dimension $p$. Let $y_d(t)$ be a bounded reference trajectory (twice differentiable). We define the tracking error as $\tilde{x}_1 = x_1 - y_d$. Applying the backstepping approach, one can design the following adaptive control law:

$$u = \ddot{y}_d + k_1\dot{y}_d - \tilde{x}_1 - k_2 z - \phi_2^T\hat{\theta} + \frac{\partial\psi}{\partial x_1}\left(x_2 + \phi_1^T\hat{\theta}\right) + \frac{\partial\psi}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{40}$$

$$\dot{\hat{\theta}} = \Gamma\left(\phi_1\tilde{x}_1 + \left(\phi_2 - \phi_1\frac{\partial\psi}{\partial x_1}\right)z\right) \tag{41}$$

where $k_1, k_2 > 0$, $\Gamma = \Gamma^T > 0$, $z = x_2 - \psi$, and

$$\psi = \dot{y}_d - k_1\tilde{x}_1 - \phi_1^T\hat{\theta}. \tag{42}$$

Under this adaptive control law, the following Lyapunov function:

$$V(\tilde{x}_1, z, \tilde{\theta}) = \frac{1}{2}\tilde{x}_1^2 + \frac{1}{2}z^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{43}$$

leads to

$$\dot{V} = -k_1\tilde{x}_1^2 - k_2 z^2. \tag{44}$$

Hence, this adaptive controller can be extended to the case where (39) is executing a repetitive task over a finite time-interval. The initial conditions at each iteration must satisfy either the resetting condition, i.e., $\tilde{x}_{1,k}(0) = z_k(0) = 0$ or the alignment condition, i.e., $\tilde{x}_{1,k}(0) = \tilde{x}_{1,k-1}(T)$, $z_k(0) = z_{k-1}(T)$. The boundedness and the convergence of $\tilde{x}_{1,k}(t)$ and $z_k(t)$ are guaranteed as per Theorem 1. In this example, a second-order system was considered for simplicity. It is worth noting that our approach can be used for a more general class of nonlinear systems stabilizable via adaptive backstepping [10]. We have just to be careful with the initial conditions since new variables are introduced at each step of the backstepping procedure. In fact, the resetting or the alignment condition has to be satisfied for all the variables, other than the parameter estimation errors, involved in the final Lyapunov function. Note that the backstepping procedure generally leads to a parametric adaptation rule with a right-hand side depending on the estimated parameters $\hat{\theta}$, and hence Remark 7 applies.

## IV. Simulation Results

We simulated Example 1, with $\theta = 1$, $k = \beta = 10$, $x_k(0) = 0$, and $x_d(t) = \sin(2\pi t)$, over the finite time interval [0,1]. Fig. 1 shows the evolution of the sup-norm of the tracking error versus the iteration number for different values of $\gamma$. Fig. 2 shows the root mean square (rms) estimation error versus the iteration number. Note that the estimated parameters converge to the real parameter for $\gamma = 0, 0.5$. In the case where $\gamma = 1$, the convergence of the estimated parameter to the real one, for all $t \in [0, 1]$, is not guaranteed.

We simulated the adaptive ILC given in Example 2 over the time interval [0, 10] s, with $a_m = -2$, $a_p = 5$, $k_m = 1$, $k_p = 1$, and $\alpha = 10$. The reference input $r(t)$ has been taken as a unit step. The initial conditions have been taken as $x_m(0) = x_k(0) = 0$, $\hat{\theta}_{1,0}(0) = \hat{\theta}_{2,0}(0) = 0$. Fig. 3 shows the evolution of the sup-norm of the tracking error versus the iteration number for different values of $\gamma$. Fig. 4 shows the rms estimation error versus the iteration number. It is clear, in this example, that the best convergence is achieved with $\gamma = 1$. The estimated parameters do not converge to the real ones, over the whole time interval, for $\gamma = 0, 0.5, 1$.

## V. Conclusion

We proposed a systematic procedure for the design of adaptive ILC schemes for uncertain nonlinear systems based on the existence of a Lyapunov function for the system under consideration. In fact, if a Lyapunov-based standard adaptive control law can be designed and the Lyapunov function satisfies properties P1) and P2), we show that the extension of the standard adaptive controller to an adaptive ILC controller is straightforward. The resulting parametric adaptation law is quite general in the sense that it includes the pure time-domain adaptation for $\gamma = 0$, the pure iteration-domain adaptation for $\gamma = 1$, and the combination of both for $\gamma \in (0, 1)$. It has been shown that the main advantages of the pure time-domain adaptation is the low memory-size requirement in real-time implementations as well as the simplicity of the design since both properties P1) and P2) are not required. The pure iteration-domain adaptation is a discrete-type integration along the iteration axis and hence does not require an approximative numerical integration at each iteration in real-time applications and does not require the unknown parameters to be time-invariant (as in the case of the pure time-domain or in the case of the combination of both adaptation types). With the pure iteration-domain adaptation, i.e., $\gamma = 1$, we guarantee the boundedness of the infinity norm of the tracking error, the boundedness of the $\mathcal{L}_2$-norm of the control input, and the convergence to zero of the $\mathcal{L}_2$-norm of the tracking error (under the alignment condition) and the convergence to zero of the infinity norm of the tracking error (under the resetting condition). With the pure time-domain or with the combination of both adaptation types, i.e., $\gamma \in [0, 1)$, we guarantee the boundedness of the infinity norm of all signals as well as the convergence to zero of the infinity norm of the tracking error.

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