

A Multichannel IOS Small Gain Theorem for Systems With Multiple Time-Varying Communication Delays

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Abstract—A version of the input-to-output stability (IOS) small gain theorem is derived for interconnections where the subsystems communicate asynchronously over multiple channels, and the communication is subject to multiple time-varying possibly unbounded communication delays. It is shown that a “multi-dimensional” version of the small gain condition guarantees the stability of the interconnection of IOS subsystems under certain mild assumptions imposed on communication process. The fulfillment of these assumptions does not depend on characteristics of the communication channels, but can always be guaranteed by implementation of certain standard features of the underlying communication protocol. This result is applicable to a wide range of dynamical systems whose parts communicate over networks, such as Internet-based teleoperators.

Index Terms—Networked control systems, small-gain condition, stability, time-varying communication delays.

I. INTRODUCTION

The small-gain theorem is possibly one of the most important results in feedback systems theory. In its simplest form, it asserts that a feedback system is stable if the product of the subsystem’s gains is less than one. After its establishment in 1960s [1], [2], the small gain theorem has found numerous applications in many areas of systems and control design. Some important references on this topic include [2]–[6], among many others (see also a survey [7] devoted to nonlinear control applications). In particular, extensions of the small gain theorem to the case of input-to-state stable (ISS, [8]) subsystems have been developed [5], [6]. In modern control applications, communication between subsystems is frequently performed over communication networks, such as local area networks (LANs) as well as the Internet. One such an application, which is of particular interest to the authors, is network-based teleoperation [9], [10]. In teleoperator systems, two (or more) manipulators called master and slave(s) are connected over a network, and the underlying control system should be designed to achieve stable and transparent teleoperation. Having a network such as the Internet as a communication medium implies existence of certain communication constraints. For example, packets in the Internet may follow different routes, and the transmission delay along each route depends on the current state of that route, existence of congestions, routers queue lengths, etc. As a result, transmission delay of each particular packet is unknown beforehand and may differ significantly from packet to packet.

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Also, occurrence of congestions may result in unbounded communication delays and packet drops. All these phenomena complicate the analysis and control of systems that include networks as a communication medium, and this fact stimulated significant research efforts over the last several years in the area of networked control systems (see [11], and bibliography therein). Moreover, in advanced teleoperator systems, the master and the slave manipulators may exchange information of different nature such as positions, velocities, forces, and video. These signals come from different sensors and may be sent asynchronously over multiple network connections which results in different delay/packet losses characteristics.

In this technical note, we present a version of the IOS Small-Gain Theorem that is designed specifically for the stability analysis of the interconnections where the communication between subsystems is performed over multiple channels, and is subject to time-varying possibly unbounded communication delays. Comparing to the well-established versions of the IOS (ISS) small gain theorem [5], [6], our work differs in at least two aspects. First, we consider multi-input-multi-output (MIMO) subsystems where a separate gain function is associated with each input-output channels pair, and address stability properties of the interconnection which is carried out over multiple channels in both the feedforward and the feedback paths. For this type of interconnections, a “multi-dimensional” small gain stability condition is established. This approach results in significantly less conservative stability criteria comparing to the traditional one, where the maximal gain over all channels are taken into consideration. In particular, in many multi-channel systems such as teleoperators, some gains but usually not all of them can be assigned arbitrarily, therefore, using the small-gain theorem based on the only one gain function that works for all the channels may lead to a wrong conclusion about impossibility to stabilize the system. Second, we formulate and prove the IOS small gain theorem for the case where the communications between subsystems is subject to unknown, time-varying and possibly unbounded communication delays, and moreover, delays in different channels may have independent characteristics. To deal with stability analysis of systems with multiple time-varying delays, we use a (multi-channel) extension of the IOS notion to systems of functional differential equations (FDEs) analogous to the one of the input-to-state stability (ISS) proposed in [12]. We prove that if both subsystems are IOS then a “multi-dimensional” version of the strict contraction (“small gain”) condition implies that the interconnected system is IOS if the communication delays satisfy certain mild assumptions. We show that the fulfillment of these assumptions does not depend on characteristics of the communication channels; on the contrary, it can always be guaranteed by implementation of certain standard features of the underlying communication protocol such as packet numbering and (or) timestamping.

References relevant to the result presented in this technical note include [13], [14]. In particular, in [13] the ISS small gain arguments are applied to the problem of stabilization of nonlinear systems in presence of quantization and bounded communication delays, while in [14] small-gain theorems for monotone dynamical systems are established that are suitable for treatment of delay-differential equations with multiple inputs and outputs. Different simplified versions of the result presented in our technical note were utilized previously in stability analysis of force-reflecting teleoperators in presence of communication delays in [15]–[17].

The technical note is organized as follows. Some preliminary materials are presented in Section II. In particular, we introduce multi-dimensional extensions of \mathcal{G} , \mathcal{K} , and \mathcal{K}_∞ -classes of functions in Section II-A, and a multi-channel version of the input-to-output stability for systems of functional-differential equations (FDEs) is defined in Section II-B. The main result together with its proof are presented in Section III. An illustrative example is given in Section IV. Finally, in Section V, some concluding remarks are given.

II. PRELIMINARIES

A. \mathcal{G} , \mathcal{K} , \mathcal{K}_∞ -Classes and Their Multidimensional Extensions

Functional classes \mathcal{G} , \mathcal{K} , and \mathcal{K}_∞ are commonly used in nonlinear control literature [18]. A continuous function $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+ := [0, +\infty)$, is said to belong to class \mathcal{G} ($\gamma \in \mathcal{G}$) if it is nondecreasing, and satisfies $\gamma(0) = 0$. A function $\gamma \in \mathcal{G}$ belongs to class \mathcal{K} ($\gamma \in \mathcal{K}$) if it is strictly increasing. Also, a function $\gamma \in \mathcal{K}$ belongs to class \mathcal{K}_∞ ($\gamma \in \mathcal{K}_\infty$) if $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$.

In this technical note, we deal with IOS stability properties of multiple inputs – multiple outputs (MIMO) systems. To simplify the notation in MIMO case, it is desirable to use an extension of the classes \mathcal{G} , \mathcal{K} , and \mathcal{K}_∞ to the case of multivariable maps $\mathbb{R}_+^m \rightarrow \mathbb{R}_+^n$, where \mathbb{R}_+^n is a set of n -tuples of nonnegative real numbers, and $m, n \in \mathbb{N}$. One possible way to define such an extension is to consider $n \times m$ -dimensional arrays of functions $\Gamma_{ij}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i = 1, 2, \dots, n, j = 1, \dots, m$. With each such an array, a map $\mathbf{\Gamma}: \mathbb{R}_+^m \rightarrow \mathbb{R}_+^n$ can be associated which is defined for each $\mathbf{s} = (s_1, s_2, \dots, s_m)^T \in \mathbb{R}_+^m$ according to the formula

$$(\mathbf{\Gamma}(\mathbf{s}))_i = \max_{j \in \{1, \dots, m\}} \Gamma_{ij}(s_j), \quad i = 1, \dots, n.$$

The map $\mathbf{\Gamma}: \mathbb{R}_+^m \rightarrow \mathbb{R}_+^n$ is said to belong to a class $\mathcal{G}^{n \times m}$ if all Γ_{ij} belong to class \mathcal{G} . Classes $\mathcal{K}^{n \times m}$ and $\mathcal{K}_\infty^{n \times m}$ are defined analogously.

Some further notations used in the technical note are as follows. Similarly to the scalar case, the composition of maps $\mathbf{\Gamma}_1 \in \mathcal{G}^{n \times m}$ and $\mathbf{\Gamma}_2 \in \mathcal{G}^{m \times p}$ is denoted by $\mathbf{\Gamma}_2 \circ \mathbf{\Gamma}_1 \in \mathcal{G}^{n \times p}$, i.e., $\mathbf{\Gamma}_2 \circ \mathbf{\Gamma}_1(\mathbf{s}) := \mathbf{\Gamma}_2(\mathbf{\Gamma}_1(\mathbf{s}))$. For vectors in \mathbb{R}_+^l , we use relations $\leq, <, \geq, >$ defined as follows: given $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}_+^l$, we will write $\mathbf{u}_1 \leq \mathbf{u}_2$ if and only if $u_1^{\{i\}} \leq u_2^{\{i\}}$ for all $i \in \{1, \dots, l\}$, and relations $<, \geq, >$ are defined analogously. Finally, the maximum of $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}_+^l$ should be understood in the elementwise sense, i.e., $\mathbf{u} := \max\{\mathbf{u}_1, \mathbf{u}_2\} \in (\mathbb{R}^+)^l$ consists of elements $u^{\{i\}} := \max\{u_1^{\{i\}}, u_2^{\{i\}}\}$, $i = 1, \dots, l$.

B. IOS for FDEs

The notion of the input-to-output stability (IOS) as well as the related notion of the input-to-state stability (ISS) were first introduced for the case of time-invariant ODEs in [19]. Since then, these notions were thoroughly developed (see for example [8] and the extensive bibliography therein). In particular, paper [12] deals with the notion of ISS for systems of functional-differential equations (FDEs). This technical note is of a special interest for our work, since we address stability properties of the interconnections where the subsystems communicate over network channels that may involve communication delays. To describe such an interconnection, a system of FDEs would be an appropriate mathematical object. Below, we use the notation borrowed from [12], [20]. Given functions $x: \mathbb{R} \rightarrow \mathbb{R}^n$, $t_d: \mathbb{R} \rightarrow \mathbb{R}_+$, denote $x_d(t) := \{x(s), s \in [t - t_d(t), t]\}$. Thus, $x_d(t)$ is a piece of trajectory which begins at $s = t - t_d(t)$ and ends at $s = t$, rather than simply its value at time t . Now, given a function $t_d: \mathbb{R} \rightarrow \mathbb{R}_+$ satisfying $t_d(t_2) - t_d(t_1) \leq t_2 - t_1$ for all $t_1, t_2 \in \mathbb{R}$, consider a system of functional differential equations where both the input and the output are partitioned into separate ‘‘channels’’, as follows

$$\begin{aligned} \dot{x}(t) &= F\left(x_d, u_d^{\{1\}}, \dots, u_d^{\{l\}}, t\right) \\ y^{\{1\}}(t) &= H^{\{1\}}\left(x_d, u_d^{\{1\}}, \dots, u_d^{\{l\}}, t\right) \\ &\vdots \\ y^{\{r\}}(t) &= H^{\{r\}}\left(x_d, u_d^{\{1\}}, \dots, u_d^{\{l\}}, t\right). \end{aligned} \quad (1)$$

Here, $u^{\{i\}} \in \mathbb{R}^{m_i}$, $i \in \{1, \dots, l\}$ are the inputs, and $y^{\{j\}} \in \mathbb{R}^{p_j}$, $j \in \{1, \dots, r\}$ are the outputs, $\sum_{i=1}^l m_i = m$, $\sum_{j=1}^r p_j = p$. It is assumed that F and $H^{\{j\}}$, $j = \{1, \dots, r\}$ are Lipschitz continuous operators in x_d, u_d uniformly for all $t \in \mathbb{R}$, and Lebesgue measurable in t . Traditional definition of the IOS deals with situation where the stability is characterized by a single IOS gain function (although benefits of using different gain functions for different inputs were mentioned, for example, in [5]). For a MIMO system of the form (1), however, it may be desirable to use a definition of IOS (ISS) where separate gain functions are specified for each pair of the input-output channels. We present such a definition using the multidimensional extension of the classes \mathcal{G} , \mathcal{K} , and \mathcal{K}_∞ introduced above. Below, we use the notation $|x_d(t)| := \sup_{s \in [t - t_d(t), t]} |x(s)|$ which is adopted from [12], and we denote $\mathbf{u}_d^+ := \left[|u_d^{\{1\}}|, \dots, |u_d^{\{l\}}|\right]^T \in \mathbb{R}_+^l$, $\mathbf{y}_d^+ := \left[|y_d^{\{1\}}|, \dots, |y_d^{\{r\}}|\right]^T \in \mathbb{R}_+^r$, where $|\cdot|$ denotes the vector ∞ -norm. A multi-channel version of the IOS notion for systems of FDEs of the form (1) can be defined as follows.

Definition 1: (IOS for FDEs) The system of the form (1) is input-to-output stable (IOS) at $t = t_0 \in \mathbb{R}$ with $t_d(t_0) \geq 0$, IOS gains $\mathbf{\Gamma} \in \mathcal{G}^{r \times l}$, restrictions $\Delta_x \in \mathbb{R}_+$, $\Delta_u \in \mathbb{R}_+^l$, and offset $\delta \in \mathbb{R}_+$ if the conditions $|x_d(t_0)| \leq \Delta_x$, and $\sup_{t \geq t_0} \mathbf{u}_d^+ \leq \Delta_u$, imply that the solution of (1) are well-defined for all $t \geq t_0$, and the following properties hold:

i) *uniform boundedness*: $\exists \beta \in \mathcal{K}_\infty^{r \times 1}$ s. t. $\sup_{t \geq t_0} \mathbf{y}^+ \leq$

$$\max \left\{ \beta(|x_d(t_0)|), \mathbf{\Gamma} \left(\sup_{t \geq t_0} \mathbf{u}_d^+ \right), \delta \right\};$$

ii) *convergence*: $\limsup_{t \rightarrow \infty} \mathbf{y}^+ \leq \max \left\{ \mathbf{\Gamma} \left(\limsup_{t \rightarrow \infty} \mathbf{u}_d^+ \right), \delta \right\}$. •

III. MAIN RESULT

Consider two systems of FDEs Σ_i , $i \in \{1, 2\}$, of the form

$$\begin{aligned} \dot{x}_i &= F_i\left(x_{id}, u_{id}^{\{1\}}, \dots, u_{id}^{\{l_i\}}, w_{id}^{\{1\}}, \dots, w_{id}^{\{v_i\}}, t\right) \\ y_i^{\{1\}} &= H_i^{\{1\}}\left(x_{id}, u_{id}^{\{1\}}, \dots, u_{id}^{\{l_i\}}, w_{id}^{\{1\}}, \dots, w_{id}^{\{v_i\}}, t\right) \\ &\vdots \\ y_i^{\{r_i\}} &= H_i^{\{r_i\}}\left(x_{id}, u_{id}^{\{1\}}, \dots, u_{id}^{\{l_i\}}, w_{id}^{\{1\}}, \dots, w_{id}^{\{v_i\}}, t\right) \end{aligned} \quad (2)$$

where the inputs are categorized into the controlled inputs u and disturbances inputs w , $r_1 = l_2 = p \in \mathbb{N}$, $r_2 = l_1 = q \in \mathbb{N}$, $v_1, v_2 \in \mathbb{N}$. As before, both $t_{id}(t)$, $i \in \{1, 2\}$, are assumed to satisfy $t_{id}(t_2) - t_{id}(t_1) \leq t_2 - t_1$ for all $t_1, t_2 \in \mathbb{R}$. In the special case where $t_{1d}(t) \equiv 0$ and/or $t_{2d}(t) \equiv 0$, the corresponding subsystem becomes a system of ODEs.

Remark 1: Our choice to describe subsystems Σ_i , $i \in \{1, 2\}$ in terms of FDEs is based on the fact that, in many cases, the subsystems that communicate over a network are implemented in sampled-data fashion. Analytically, a sampled-data nonlinear system can be described either in continuous time as a system of FDEs (more precisely, as a system of retarded differential equations with time-varying uniformly bounded delays) or represented by its discrete-time model (see [21]). Although both these approaches have their merits and advantages, the continuous-time approach seems to be preferable over the discrete-time one in at least two aspects. First, modelling of a continuous-time nonlinear plant in the discrete-time domain requires exact integration of the corresponding differential equations, which is generally not possible for a nonlinear system. Second, the continuous-time approach appears to be more complete, because the continuous-time plant always exhibits some inter-sampling dynamics which are ignored in the discrete-time approach. In particular, stability of a discrete-time model does not necessarily imply stability of the corresponding sampled-data

system, since the intersampling behaviour of a continuous-time plant may exhibit peaking (see, for example [22]). These are two reasons that might justify our choice of a continuous-time framework. •

The stability properties of the interconnection of Σ_1 and Σ_2 are addressed in this technical note under the assumption that both the subsystems are IOS. Below, this assumption is presented formally.

Assumption 1: The systems Σ_1, Σ_2 are IOS with restrictions $\Delta_{x_i}, \Delta_{u_i}, \Delta_{w_i}$ and offsets $\delta_i, i \in \{1, 2\}$, respectively, where $\Delta_{x_1}, \Delta_{x_2} \in \mathbb{R}_+, \Delta_{u_1} \in \mathbb{R}_+^q, \Delta_{u_2} \in \mathbb{R}_+^p, \delta_1 \in \mathbb{R}_+^p, \delta_2 \in \mathbb{R}_+^q$, and $\Delta_{w_i} \in \mathbb{R}_+^{v_i}, i \in \{1, 2\}$. More precisely, there exist $\beta_1 \in \mathcal{K}_\infty^{p \times 1}, \beta_2 \in \mathcal{K}_\infty^{q \times 1}, \mathbf{\Gamma}_{1u} \in \mathcal{G}^{p \times q}, \mathbf{\Gamma}_{2u} \in \mathcal{G}^{q \times p}, \mathbf{\Gamma}_{1w} \in \mathcal{G}^{p \times v_1}$, and $\mathbf{\Gamma}_{2w} \in \mathcal{G}^{q \times v_2}$, such that for each $i \in \{1, 2\}$ and each $t_0 \in \mathbb{R}$, the conditions $|x_i(t_0)| \leq \Delta_{x_i}, \sup_{t \geq t_0} \mathbf{u}_i^+ \leq \Delta_{u_i}$, and $\sup_{t \geq t_0} \mathbf{w}_i^+ \leq \Delta_{w_i}$ imply that the corresponding solution of Σ_i is well-defined for all $t \in [t_0, +\infty)$, and the following inequalities hold

$$\sup_{t \geq t_0} \mathbf{y}_i^+ \leq \max \left\{ \beta_i(|x_{id}(t_0)|), \sup_{t \geq t_0} \mathbf{\Gamma}_{iu}(\mathbf{u}_{id}^+), \sup_{t \geq t_0} \mathbf{\Gamma}_{iw}(\mathbf{w}_{id}^+), \delta_i \right\} \quad (3)$$

$$\limsup_{t \rightarrow +\infty} \mathbf{y}_i^+ \leq \max \left\{ \limsup_{t \rightarrow +\infty} \mathbf{\Gamma}_{iu}(\mathbf{u}_{id}^+), \limsup_{t \rightarrow +\infty} \mathbf{\Gamma}_{iw}(\mathbf{w}_{id}^+), \delta_i \right\}. \quad (4)$$

Remark 2: The form of Assumption 1, in particular, allows the subsystem to be implemented as a sampled-data system with quantization. Indeed, for a nonlinear sampled-data system with quantization, the existing design methods (such as the controller emulation approach as well as the design based on approximate discrete-time models) generally lead to stability with finite restriction and nonzero offset [21], which meets the requirements imposed by Assumption 1. •

Below, we consider the interconnection of Σ_1, Σ_2 where the communication between the subsystems is performed over multiple communication channels which are subject to constraints typical for serial communication networks, such as existence of unknown time-varying possibly unbounded communication delays as well as possible packet losses. More precisely, consider the interconnection of Σ_1, Σ_2 whose inputs and outputs are related according to the formulas

$$\mathbf{u}_1^+(t) \equiv 0, \quad \mathbf{u}_2^+(t) \equiv 0 \quad \text{for } t < T_0 \quad (5)$$

$$\mathbf{u}_1^+(t) \leq \mathbf{\Psi}_2(\hat{\mathbf{y}}_2^+(t)), \quad \mathbf{u}_2^+(t) \leq \mathbf{\Psi}_1(\hat{\mathbf{y}}_1^+(t)) \quad \text{for } t \geq T_0 \quad (6)$$

where $T_0 \in \mathbb{R}$ is the time instant when a connection between the subsystems has been established, $\mathbf{\Psi}_1 \in \mathcal{G}^{p \times p}, \mathbf{\Psi}_2 \in \mathcal{G}^{q \times q}$, and $\hat{\mathbf{y}}_1^+, \hat{\mathbf{y}}_2^+$ are delayed versions of $\mathbf{y}_1^+, \mathbf{y}_2^+$, defined as follows

$$\hat{\mathbf{y}}_1^+(t) = \left[|y_1^{\{1\}}(t - \tau_f^{\{1\}}(t))|, \dots, |y_1^{\{p\}}(t - \tau_f^{\{p\}}(t))| \right]^T$$

$$\hat{\mathbf{y}}_2^+(t) = \left[|y_2^{\{1\}}(t - \tau_b^{\{1\}}(t))|, \dots, |y_2^{\{q\}}(t - \tau_b^{\{q\}}(t))| \right]^T. \quad (7)$$

The assumption below is imposed on communication delays $\tau_f^{\{i\}}, \tau_b^{\{j\}}$.

Assumption 2: The communication delays $\tau_f^{\{i\}}, \tau_b^{\{j\}}: [T_0, +\infty) \rightarrow \mathbb{R}_+, i \in \{1, \dots, p\}, j \in \{1, \dots, q\}$, are Lebesgue measured functions with the following properties:

- i) there exist $\tau_* > 0$ and a piecewise continuous function $\tau^*: \mathbb{R} \rightarrow \mathbb{R}_+$ satisfying $\tau^*(t_2) - \tau^*(t_1) \leq t_2 - t_1$, such that the following inequalities hold for all $t \geq 0$

$$\tau_* \leq \min_{\substack{i \in \{1, \dots, p\} \\ j \in \{1, \dots, q\}}} \left\{ \tau_f^{\{i\}}(t), \tau_b^{\{j\}}(t) \right\}$$

$$\leq \max_{\substack{i \in \{1, \dots, p\} \\ j \in \{1, \dots, q\}}} \left\{ \tau_f^{\{i\}}(t), \tau_b^{\{j\}}(t) \right\} \leq \tau^*(t); \quad (8)$$

- ii)

$$t - \max_{\substack{i \in \{1, \dots, p\} \\ j \in \{1, \dots, q\}}} \left\{ \tau_f^{\{i\}}(t), \tau_b^{\{j\}}(t) \right\} \rightarrow +\infty \quad \text{as } t \rightarrow +\infty. \quad (9)$$

Assumption 2 is not a restrictive one. In particular, part i) implies the existence of an upper bound $\tau^*(t)$ for delays $\tau_f^{\{i\}}(t), \tau_b^{\{j\}}(t)$, which is possibly a time-varying unbounded function of time that does not grow faster than the time itself. Note that the lower bound $\tau_* > 0$ always exists in any practical (not necessarily networked-based) feedback system due to natural transmission/propagation constraints. From mathematical point of view, however, the existence of a positive lower bound is a purely technical assumption which allows to simplify formulation and proof of the small-gain theorem, in particular, to avoid dealing with well-posedness issue. Assumption 2, i) can always be satisfied in real world communication networks by using standard techniques such as sequence numbering and (or) timestamping. For example, a sequence number can be assigned to each outgoing packet at the sender side, while at the receiver side, the control signals are applied to the system's input in order according to their packet's sequence number. In this case, the packets with sequence numbers less than those already applied to the system's input are considered to be outdated and has to be discarded. As a result, the control signals are applied to the input in the same order as they were sent (possibly with some packets lost). Mathematically, this implies that $t - \tau(t)$, where $\tau(\cdot)$ is the corresponding communication delay, is a nondecreasing function of time t . Now, suppose we have n communication channels with the corresponding communication delays $\tau_i(\cdot), i \in \{1, \dots, n\}$, such that $t - \tau_i(t)$ are nondecreasing functions of time. Denote $\tau_{\max}(\cdot) = \max_i \tau_i(\cdot)$, then $\min_i (t - \tau_i(t)) = t - \tau_{\max}(t)$ is also a nondecreasing function of time, i.e., $t_1 - \tau_{\max}(t_1) \leq t - \tau_{\max}(t)$ holds for all $t \geq t_1$. Rewriting the last inequality, we see that $\tau_{\max}(t) - \tau_{\max}(t_1) \leq t - t_1$ holds for all $t \geq t_1$, i.e., the Assumption 2, i) is satisfied with an upper bound $\tau^*(\cdot) := \tau_{\max}(\cdot)$.

On the other hand, Assumption 2 can also be satisfied by using timestamps and by assigning a maximal packet lifetime $T_{\max} < \infty$, so that all packets "older" than T_{\max} are to be discarded on the receiver's side. In this case, Assumption 2, i) holds with $\tau^*(t) := t - T_0 + \tau_{\max}(T_0) + T_{\max}$, where $\tau_{\max}(\cdot) := \max_i \tau_i(\cdot)$. To prove this, pick an arbitrary $t \geq T_0$. Suppose there are no packets arrived to the receiver over i -th channel during the time interval $[T_0, t]$, then we have $\tau_i(t) \leq t - T_0 + \tau_{\max}(T_0)$. Otherwise, let $t_1 \in [T_0, t]$ be the last time instant (within the interval $[T_0, t]$) when a new packet arrives to the receiver over i -th channel. Then, $\tau_i(t_1) \leq T_{\max}$, which implies that $\tau_i(t) \leq t - t_1 + T_{\max} \leq t - T_0 + T_{\max}$. In both these cases, we have $\tau_i(t) \leq t - T_0 + \tau_{\max}(T_0) + T_{\max} := \tau^*(t)$. Clearly, the function $\tau^*(\cdot)$ satisfies $\tau^*(t_2) - \tau^*(t_1) = t_2 - t_1$, i.e., $\tau^*(\cdot)$ is an upper bound that satisfies Assumption 2, i).

Finally, for the above described schemes with time-stamping and (or) sequence numbering, part ii) of Assumption 2 simply means the existence of communication between the subsystems on a semi-infinite

time interval. To clarify this statement, assume the converse, *i.e.*, there exists a sequence $s_j \rightarrow +\infty, j = 1, 2, \dots$, and $t_L < +\infty$ such that

$$s_j - \tau_i(s_j) \leq t_L \quad (10)$$

holds for some $i \in \{1, \dots, n\}$ and for all s_j . For the scheme with sequence numbering, it is shown above that $t - \tau_i(t)$ is a nondecreasing function of t , therefore, (10) is equivalent to $\sup_{t \geq T_0} \{t - \tau_i(t)\} \leq t_L$. The last simply implies that the packets which are sent after t_L would never be received by the other subsystem, *i.e.*, no communication exists between the subsystems during the time interval (t_L, ∞) . For the scheme with timestamping, on the other hand, it is shown above that if a packet arrives to the receiver at some instant $t_{arr} \in [T_0, +\infty)$ then

$$\tau_i(s_j) \leq s_j - t_{arr} + T_{\max} \quad (11)$$

holds for all $s_j \geq t_{arr}$. Combining (10) and (11), we see that $t_{arr} \leq t_L + T_{\max}$, *i.e.*, no packets would arrive to the receiver after $t = t_L + T_{\max}$. Overall, Assumption 2, ii) is always satisfied unless the communication between the subsystems is totally lost on a semi-infinite time interval.

Below, we use the notation $\mathbf{x}^+ := (|x_1|, |x_2|)^T \in \mathbb{R}_+^2$, $\mathbf{y}^+ := (\mathbf{y}_1^{+T}, \mathbf{y}_2^{+T})^T \in \mathbb{R}_+^{p+q}$, and $\mathbf{w}^+ := (\mathbf{w}_1^{+T}, \mathbf{w}_2^{+T})^T \in \mathbb{R}_+^{v_1+v_2}$. The following is the main result of this technical note.

Theorem 1: Suppose the system (2)–(7) satisfies Assumptions 1, 2, and there exist $\delta, \Delta \in \mathbb{R}_+^p$, $\mathbf{0} \leq \delta < \Delta$, such that the following small gain condition holds

$$\Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2u} \circ \Psi_1(\mathbf{s}) < \mathbf{s} \quad \text{for all } \mathbf{s} \in (\delta, \Delta). \quad (12)$$

If $\Delta > \Delta^*$, where

$$\Delta^* := \max \left\{ \begin{array}{l} \beta_1(\Delta_{x1}), \Gamma_{1u} \circ \Psi_2 \circ \beta_2(\Delta_{x2}), \delta \\ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w}(\Delta_{w2}), \Gamma_{1w}(\Delta_{w1}^+) \end{array} \right\} \quad (13)$$

then the system (2)–(7), is IOS at $t = T_0$ in the sense of Definition 1 with

$$t_d(T_0) := t_{1d}(T_0) + t_{2d}(T_0) + \tau^*(T_0) + \tau^*(T_0 - \tau^*(T_0)) \quad (14)$$

restrictions $\Delta_x := (\Delta_{x1}, \Delta_{x2})^T \in \mathbb{R}_+^2$, $\Delta_w := (\Delta_{w1}^T, \Delta_{w2}^T)^T \in \mathbb{R}_+^{v_1+v_2}$, and offset $\delta_0 := (\delta^T, (\Gamma_{2u} \circ \Psi_1(\delta))^T)^T \in \mathbb{R}_+^{p+q}$. More precisely, the conditions $\mathbf{x}_d^+(T_0) \leq \Delta_x$, $\sup_{t \geq T_0} \mathbf{w}_d^+ \leq \Delta_w$ imply that the following inequalities hold

$$\sup_{t \geq T_0} \mathbf{y}^+ \leq \max \left\{ \beta(\mathbf{x}_d^+(T_0)), \Gamma_w \left(\sup_{t \geq T_0} \mathbf{w}_d^+ \right), \delta_0 \right\} \quad (15)$$

$$\limsup_{t \rightarrow +\infty} \mathbf{y}^+ \leq \max \left\{ \Gamma_w \left(\limsup_{t \rightarrow +\infty} \mathbf{w}_d^+ \right), \delta_0 \right\} \quad (16)$$

where $\beta \in \mathcal{G}^{(p+q) \times 2}$ and $\Gamma_w \in \mathcal{G}^{(p+q) \times (v_1+v_2)}$ are defined as follows

$$\beta := \begin{bmatrix} \beta_1 & \Gamma_{1u} \circ \Psi_2 \circ \beta_2 \\ \Gamma_{2u} \circ \Psi_1 \circ \beta_1 & \beta_2 \end{bmatrix} \\ \Gamma_w := \begin{bmatrix} \Gamma_{1w} & \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w} \\ \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1w} & \Gamma_{2w} \end{bmatrix}. \quad (17)$$

Proof of Theorem 1

Consider the system (2)–(7), and suppose

$$\begin{aligned} |x_{1d}(T_0)| \leq \Delta_{x1}, |x_{2d}(T_0)| \leq \Delta_{x2} \\ \sup_{t \in [T_0, +\infty)} |\mathbf{w}_{1d}^+| \leq \Delta_{w1}, \sup_{t \in [T_0, +\infty)} |\mathbf{w}_{2d}^+| \leq \Delta_{w2} \end{aligned} \quad (18)$$

where $\Delta_{x1}, \Delta_{x2}, \Delta_{w1}, \Delta_{w2}$ satisfy (13). Since $\mathbf{u}_1^+(t) \equiv \mathbf{0}$, and $\mathbf{u}_2^+(t) \equiv \mathbf{0}$ for $t < T_0$, Assumption 1 together with (18) as well as causality arguments imply that

$$\begin{aligned} \mathbf{y}_{id}^+(T_0) \leq \max \{ \beta_i(|x_{id}(T_0)|), \Gamma_{iw}(\mathbf{w}_{id}^+(T_0)), \delta_i \} \\ \leq \max \{ \beta_i(\Delta_{xi}), \Gamma_{iw}(\Delta_{wi}), \delta_i \} \end{aligned} \quad (19)$$

holds for $i \in \{1, 2\}$. Taking into account (5), (6), as well as Assumption 2, i), we see that $\sup_{t \in [T_0 - t_d(T_0), T_0 + \tau_*]} \mathbf{u}_1^+ \leq \Psi_2(\max \{ \beta_2(\Delta_{x2}), \Gamma_{2w}(\Delta_{w2}) \})$, and $\sup_{t \in [T_0 - t_d(T_0), T_0 + \tau_*]} \mathbf{u}_2^+ \leq \Psi_1(\max \{ \beta_1(\Delta_{x1}), \Gamma_{1w}(\Delta_{w1}) \})$, *i.e.*, all the inputs of the system (2)–(7) are uniformly essentially bounded on $[T_0 - t_d(T_0), T_0 + \tau_*]$. Due to the IOS assumptions imposed on the subsystems, this implies that the solutions of (2)–(7) is well-defined for all $t \in [T_0, T_{\max}]$, where $T_{\max} > T_0 + \tau_* > T_0$. Furthermore, the following inequality holds

$$\sup_{t \in [T_0, T_{\max}]} \mathbf{y}_{1d}^+ \leq \Delta^* \quad (20)$$

where $\Delta^* < \Delta$ is defined by (13). To prove (20), assume the converse. Then there exists $T_1 \in [T_0, T_{\max} - \tau_*)$ such that

$$\sup_{t \in [T_0, T_1]} \mathbf{y}_{1d}^+ \leq \Delta^*, \quad \text{and} \quad \sup_{t \in [T_0, T_1 + \tau_*]} \mathbf{y}_1^+ > \Delta^*. \quad (21)$$

However, from (3), (14), and (18), together with Assumption 2, i), we see that

$$\sup_{t \in [T_0, T_0 + \tau_*]} \mathbf{y}_1^+ \leq \max \left\{ \begin{array}{l} \beta_1(\Delta_{x1}), \Gamma_{1w}(\Delta_{w1}) \\ \Gamma_{1u} \circ \Psi_2 \circ \beta_2(\Delta_{x2}) \\ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w}(\Delta_{w2}) \\ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2u} \circ \Psi_1 \left(\sup_{t \in [T_0, T_1]} \mathbf{y}_{1d}^+ \right) \end{array} \right\}.$$

Taking into account (12), (13), and the first inequality in (21), we get

$$\sup_{t \in [T_0, T_1 + \tau_*]} \mathbf{y}_1^+ \leq \max \left\{ \begin{array}{l} \beta_1(\Delta_{x1}), \Gamma_{1w}(\Delta_{w1}) \\ \Gamma_{1u} \circ \Psi_2 \circ \beta_2(\Delta_{x2}) \\ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w}(\Delta_{w2}), \Delta_* \end{array} \right\} := \Delta_* \quad (22)$$

which contradicts the second inequality in (21). This contradiction proves (20).

Next, let us show that $T_{\max} = +\infty$. To prove this, assume the converse, *i.e.*, $T_{\max} < +\infty$. Since both the subsystems are assumed to be IOS, the last implies that $\sup_{t \in [T_0, T_{\max}]} \max \{ \mathbf{u}_1^+(t), \mathbf{u}_2^+(t) \} = +\infty$, which may be possible only if

$$\sup_{t \in [T_0, T_{\max} - \tau_*]} \max \{ \mathbf{y}_1^+(t), \mathbf{y}_2^+(t) \} = +\infty. \quad (23)$$

Combining (3) with (5), (6), it is easy to see that (23) necessarily implies that $\sup_{t \in [T_0, T_{\max} - \tau_*]} \mathbf{y}_1^+(t) = +\infty$, which contradicts (20). Thus, $T_{\max} = +\infty$. In particular, (20) becomes

$$\sup_{t \in [T_0, +\infty)} \mathbf{y}_{1d}^+ \leq \Delta^* \quad (24)$$

Now, the proof can be easily finalized as follows. Combining (3) with (6)–(7), (19), and using Assumption 2, i), one gets

$$\sup_{t \geq T_0} \mathbf{y}_{1d}^+ \leq \max \left\{ \begin{array}{l} \beta_1(|x_{1d}(T_0)|), \Gamma_{1w} \left(\sup_{t \geq T_0} \mathbf{w}_{1d}^+(t) \right) \\ \Gamma_{1u} \circ \Psi_2 \circ \beta_2(|x_{2d}(T_0)|) \\ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w} \left(\sup_{t \geq T_0} \mathbf{w}_{2d}^+(t) \right) \\ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2u} \circ \Psi_1 \left(\sup_{t \geq T_0} \mathbf{y}_{1d}^+ \right) \end{array} \right\} \quad (25)$$

and taking into account (24) and the small-gain condition (12), it is easy to see that

$$\sup_{t \geq T_0} \mathbf{y}_{1d}^+ \leq \max \left\{ \begin{array}{l} \beta_1(|x_{1d}(T_0)|), \Gamma_{1w} \left(\sup_{t \geq T_0} \mathbf{w}_{1d}^+(t) \right) \\ \Gamma_{1u} \circ \Psi_2 \circ \beta_2(|x_{2d}(T_0)|) \\ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w} \left(\sup_{t \geq T_0} \mathbf{w}_{2d}^+(t) \right), \delta \end{array} \right\}. \quad (26)$$

On the other hand, combining (3), (5), (6) with (26), and taking into account (19), we get

$$\sup_{t \geq T_0} \mathbf{y}_2^+ \leq \max \left\{ \begin{array}{l} \beta_2(|x_{2d}(t_0)|), \Gamma_{2u} \circ \Psi_1 \circ \beta_1(|x_{1d}(T_0)|) \\ \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1u} \circ \Psi_2 \circ \beta_2(|x_{2d}(T_0)|) \\ \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1w} \left(\sup_{t \geq T_0} \mathbf{w}_{1d}^+ \right) \\ \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w} \left(\sup_{t \geq T_0} \mathbf{w}_{2d}^+ \right) \\ \Gamma_{2w} \left(\sup_{t \geq T_0} \mathbf{w}_{2d}^+ \right) \end{array} \right\}.$$

Under the assumptions of the theorem, however, we have

$$\begin{aligned} \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1u} \circ \Psi_2 \circ \beta_2(|x_{2d}(t_0)|) \\ \leq \max \{ \beta_2(|x_{2d}(t_0)|), \Gamma_{2u} \circ \Psi_1(\delta) \}. \end{aligned} \quad (27)$$

Indeed, note first that the small gain condition (12) imply

$$\Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1u} \circ \Psi_2(\mathbf{s}) < \mathbf{s} \quad (28)$$

for all $\mathbf{s} \in (\Gamma_{2u} \circ \Psi_1(\delta), \Gamma_{2u} \circ \Psi_1(\Delta))$. Also, (13) imply that

$$\Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1u} \circ \Psi_2 \circ \beta_2(|x_{2d}(t_0)|) \leq \Gamma_{2u} \circ \Psi_1(\Delta^*) := \Delta_1^*. \quad (29)$$

Now, if $\beta_2(|x_{2d}(t_0)|) \leq \Delta_1^*$, then (27) follows from (28). Otherwise (i.e., if $\beta_2(|x_{2d}(t_0)|) > \Delta_1^*$), the inequality (27) follows from (29). Analogously,

$$\begin{aligned} \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1u} \circ \Psi_2 \circ \Gamma_{2w} \left(\sup_{t \geq T_0} \mathbf{w}_{2d}^+ \right) \\ \leq \max \left\{ \Gamma_{2w} \left(\sup_{t \geq T_0} \mathbf{w}_{2d}^+ \right), \Gamma_{2u} \circ \Psi_1(\delta) \right\}. \end{aligned} \quad (30)$$

Taking into account (27), (30), we get

$$\sup_{t \geq T_0} \mathbf{y}_2^+ \leq \max \left\{ \begin{array}{l} \beta_2(|x_{2d}(T_0)|), \Gamma_{2u} \circ \Psi_1(\delta) \\ \Gamma_{2u} \circ \Psi_1 \circ \beta_1(|x_{1d}(T_0)|) \\ \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1w} \left(\sup_{t \geq T_0} \mathbf{w}_{1d}^+ \right) \\ \Gamma_{2w} \left(\sup_{t \geq T_0} \mathbf{w}_{2d}^+ \right) \end{array} \right\}. \quad (31)$$

Finally, combining (26) and (31), and using the notation (17), we get (15).

To prove convergence, note that Assumption 2, ii), implies that

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \mathbf{u}_1^+ &\leq \limsup_{t \rightarrow +\infty} \Psi_2(\mathbf{y}_2^+) \\ \limsup_{t \rightarrow +\infty} \mathbf{u}_2^+ &\leq \limsup_{t \rightarrow +\infty} \Psi_1(\mathbf{y}_1^+), \end{aligned} \quad (32)$$

Substituting the above inequalities into (4) and taking into account (12), (24), we get

$$\limsup_{t \rightarrow +\infty} \mathbf{y}_2^+ \leq \max \left\{ \begin{array}{l} \Gamma_{2u} \circ \Psi_1 \circ \Gamma_{1w} \left(\limsup_{t \rightarrow +\infty} \mathbf{w}_{1d}^+ \right) \\ \Gamma_{2w} \left(\limsup_{t \rightarrow +\infty} \mathbf{w}_{2d}^+ \right), \Gamma_{2u} \circ \Psi_1(\delta) \end{array} \right\}.$$

The convergence of \mathbf{y}_2^+ can be shown analogously. This completes the proof of Theorem 1. \bullet

IV. EXAMPLE

To illustrate applicability of the result presented, let us present the following very simple example. Consider a teleoperator system which consists of a master and a slave manipulators connected through a communication channel. A human operator moves the master, and the information about the master trajectory is sent to the remotely located slave. The slave is designed to follow the motion of the master. When slave is in contact with the environment, the information about contact forces is sent back to the master and applied to master's motor to make the human operator feel the interaction. Suppose the master manipulator is equipped with a local "PD+gravity compensation" control algorithm; the closed-loop master subsystem is then described by the following equations

$$H_m(q_m) \ddot{q}_m + C_m(q_m, \dot{q}_m) \dot{q}_m + K_m(\dot{q}_m + \Lambda_m q_m) = f_h + \hat{f}_e$$

where q_m is the position of the master, $H_m(q_m)$, $C_m(q_m, \dot{q}_m)$ are the matrices of inertia and Coriolis/centrifugal forces, $K_m = K_m^T > 0$, $\Lambda_m = \Lambda_m^T > 0$ are controller parameters, f_h is the force applied by the human, and \hat{f}_e is the force reflection term. It is easy to check (for details, see [16]) that the system is ISS, and the gain from f_h to $(q_m^T, \dot{q}_m^T)^T$ (denoted by $\gamma_{[f_h - (q_m, \dot{q}_m)]} \in \mathcal{K}$) can be made "arbitrarily small" by an appropriate choice of K_m , Λ_m ; however, the gain from \hat{f}_e to the "output" \ddot{q}_m (denoted by $\gamma_{[\hat{f}_e - \ddot{q}_m]} \in \mathcal{K}$) cannot be assigned arbitrarily. On the other hand, let the "slave+environment" interconnection be described as follows

$$\dot{x}_s = F_s(x_s, \hat{q}_m, \hat{\dot{q}}_m, \hat{\ddot{q}}_m), \quad f_e = G_s(x_s, \hat{q}_m, \hat{\dot{q}}_m, \hat{\ddot{q}}_m)$$

where x_s is a state of the "slave+environment" subsystem, $\hat{q}_m, \hat{\dot{q}}_m, \hat{\ddot{q}}_m$ are delayed versions of the master position, velocity, and acceleration, and f_e is the contact force due to environment which plays the role of the output of the subsystem; f_e is then sent over the communication channel, and its delayed version \hat{f}_e is applied to the motors of the master. (Normally, $\hat{\dot{q}}_m, \hat{\ddot{q}}_m$ are recovered from \hat{q}_m using some sort of estimation/filtering process rather than directly sent over the communication channel, but this is not important for our purposes). Suppose the control system is designed on the slave side such that the "slave+environment" subsystem is ISS. In most situations, the system can be designed such that the gain from \hat{q}_m to f_e (denoted by $\gamma_{[\hat{q}_m - f_e]} \in \mathcal{K}$) can be made negligible (arbitrarily small), however, the gain $\gamma_{[(\hat{q}_m, \hat{\dot{q}}_m) - f_e]} \in \mathcal{K}$ significantly depends on environmental damping/stiffness, and cannot be made small without deterioration of the slave's tracking properties. For this system, using traditional

approach (one gain that works for all input/output pairs) one gets the following small gain condition

$$\max\{\gamma_{[(\hat{q}_m, \dot{q}_m) \rightarrow f_e]}, \gamma_{[\hat{q}_m \rightarrow f_e]}\} \circ \max\{\gamma_{[f_e \rightarrow (q_m, \dot{q}_m)]}, \gamma_{[f_e \rightarrow \dot{q}_m]}\}(s) < s,$$

which is not satisfied if $\gamma_{[(\hat{q}_m, \dot{q}_m) \rightarrow f_e]} \circ \gamma_{[f_e \rightarrow \dot{q}_m]}$ is not a strict contraction. Thus, in this case the traditional form of the IOS small gain theorem does not guarantee stability. On the other hand, using “multi-channel” approach proposed, one gets the small gain condition of the form

$$\left[\begin{array}{c} \gamma_{[(\hat{q}_m, \dot{q}_m) \rightarrow f_e]} \quad \gamma_{[\hat{q}_m \rightarrow f_e]} \\ \gamma_{[f_e \rightarrow (q_m, \dot{q}_m)]} \quad \gamma_{[f_e \rightarrow \dot{q}_m]} \end{array} \right] (s) \\ := \max\{\gamma_{[(\hat{q}_m, \dot{q}_m) \rightarrow f_e]} \circ \gamma_{[f_e \rightarrow (q_m, \dot{q}_m)]}, \gamma_{[\hat{q}_m \rightarrow f_e]} \circ \gamma_{[f_e \rightarrow \dot{q}_m]}\}(s) \\ < s$$

which can always be satisfied (possibly with some restriction and offset) by choosing $\gamma_{[f_e \rightarrow (q_m, \dot{q}_m)]}, \gamma_{[\hat{q}_m \rightarrow f_e]} \in \mathcal{K}$ “sufficiently small.” Thus, the multi-channel version of the small gain theorem guarantees stability, while the traditional version does not.

V. CONCLUSION

In this technical note, we have presented a version of the IOS small gain theorem that is designed specifically for the stability analysis of interconnections where communication between subsystems is performed asynchronously over multiple channels and is subject to constraints typical for communication networks. In particular, it allows to handle multiple time-varying possibly discontinuous and unbounded communication delays. The assumptions imposed on communication delays can always be satisfied in real world communication networks using techniques such as timestamping and (or) sequence numbering. Although this result mainly targets stability analysis and control design of the Internet-based multichannel teleoperator systems, it is also suitable for a wide range of other distributed applications whose parts communicate over networks such as the Internet.

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