Attitude Synchronization of Multiple Rigid Bodies With Communication Delays

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***Abstract*—We consider the attitude synchronization problem of multiple rigid bodies (or spacecraft) in the presence of communication delays. Specifically, we propose a virtual systems-based approach that removes the requirement of the angular velocity measurements. First, we present a solution to the leaderless and leader-follower problems in the case of time-varying communication delays and undirected communication topology. Second, we present an attitude synchronization scheme that solves the leaderless problem under directed interconnection between rigid bodies in the presence of constant communication delays. Finally, we show that the proposed schemes can be extended in a straightforward manner to solve the cooperative attitude tracking control problem.**

***Index Terms*—Attitude synchronization, communication delays, output feedback.**

I. Introduction

The attitude synchronization problem of rigid bodies or multiple spacecraft has gained extensive interest in recent years. In deep space applications for instance, this interest is motivated by the advantages gained by replacing a traditionally large and expensive spacecraft by a cluster of micro-satellites to accomplish a common task in a coordinated manner [1]. Several solutions to the attitude synchronization problem in the full state information case can be found in, e.g., [2], [3], and references therein. Some solutions to this problem, in the case where the angular velocities are not available for feedback, have also been reported in the literature (e.g., [4]–[7]). In these papers, information exchange between members of the team plays a central role, however, communication delays that are inherently present in transmission systems have not been considered.

The effects of communication delays in linear multi-agent systems, described by second-order dynamics, have been extensively studied in [8]–[10] to cite only a few, and sufficient conditions have been derived to achieve the stability of the system. The communication delays in nonlinear systems have also been considered in bilateral teleoperation [11], [12], synchronization of Euler-Lagrange systems [13], [14], and the formation control of a class of unmanned aerial vehicles [15]. However, it is not straightforward to extend the results of the above papers to the attitude synchronization problem. The main difficulty resides in the nonlinearity of the attitude dynamics, where the angular velocity of the rigid body cannot be integrated to obtain an equivalent orientation variable. This is the reason behind the existence of only few papers dealing with this problem in the available literature.

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Using the Modified Rodriguez parameters (MRP) and the Lagrangian formulation for the attitude dynamics, the authors in [16] proposed a solution to the spacecraft attitude synchronization problem in the presence of constant communication delays. To avoid the inherent singularity of the MRP representation, the globally non-singular unit-quaternion representation has been considered in [17] and a variable structure attitude synchronization scheme for a team of spacecraft is proposed in the presence of time-varying communication delays. In [18], a similar control problem is addressed using continuous control laws. The proposed control schemes in the above papers rely on some synchronization variables defined in terms of both attitude and angular velocity tracking errors. More recently, a different analysis method has been considered in [19], where the relative attitudes are defined using linear differences between individual attitudes given in terms of unit-quaternion. It is worth mentioning that in [16]–[19], only the cooperative attitude tracking problem has been considered, where a common desired attitude trajectory is required to be available to each spacecraft in the team. Moreover, the communication topology between spacecraft is assumed to be undirected. In [20], the attitude kinematics of rigid bodies have been considered to design appropriate angular velocity inputs to solve the leaderless attitude synchronization problem with delayed communication and directed communication topologies. However, the attitude dynamics have not been considered and the input torque that drives this type of systems has not been designed. In addition, all the aforementioned papers rely on the assumption that the angular velocities are available for feedback.

The main contribution of this paper is to propose new quaternion-based attitude synchronization schemes for a group of rigid bodies (or spacecraft) in the presence of communication delays. In particular, we consider the case where the angular velocities of the rigid bodies are not available for feedback. As mentioned earlier, some solutions to this problem exist in the case of no communication delays. However, it is generally difficult to study the effects of delayed communication on these schemes, using Lyapunov-Krasovskii functionals for example. To solve this problem, we propose a virtual systems approach that handles communication delays with the missing angular velocities of rigid bodies. Using this approach, we present first a unified scheme that solves the leaderless and leader follower attitude synchronization problems in the presence of time-varying communication delays. We derive sufficient conditions on the communication delays and the controller gains such that the control objectives are attained under a fixed and undirected communication topology. Then, we present a solution to the leaderless problem in the case of constant communication delays and directed communication topologies. In each of the above cases, the extension of the proposed schemes to the full state information case is explained. To the best of our knowledge, there is no complete solution to the leaderless attitude synchronization problem with delayed communication in the full state information case, and the leader follower problem has not been addressed in this case. Moreover, we will show that the proposed schemes can be slightly modified to solve the cooperative attitude tracking problem. This extension complements the available literature by providing angular velocity-free solutions to this problem in the case of directed communication topologies.

II. Preliminaries

Throughout the paper, we omit the arguments of time-dependent signals, (e.g., $\mathbf{Q}_i \leftrightarrow \mathbf{Q}_i(t)$), except for those which are time-delayed (e.g., $\mathbf{Q}_i(t-\tau_{ij})$). In addition, the argument of the signals inside the integrals is omitted, which is assumed to be equal to the variable on the differential, unless otherwise stated (e.g. $\int_0^t \dot{\mathbf{Q}}_i ds \leftrightarrow \int_0^t \dot{\mathbf{Q}}_i(s) ds$).

A. System Model

Consider a group of n-rigid bodies, where the equations of motion of the i^{th} rigid body are given by

$$\dot{\mathbf{Q}}_i = \frac{1}{2}\mathbf{T}(\mathbf{Q}_i)\boldsymbol{\omega}_i, \qquad \mathbf{I}_{f_i}\dot{\boldsymbol{\omega}}_i = \boldsymbol{\Gamma}_i - \mathbf{S}(\boldsymbol{\omega}_i)\mathbf{I}_{f_i}\boldsymbol{\omega}_i \tag{1}$$

for $i \in \mathcal{N} := \{1, \ldots, n\}$, where $\boldsymbol{\omega}_i \in \mathbb{R}^3$ is the angular velocity of the i^{th} rigid body expressed in the body-fixed frame, $\mathcal{F}_i$, $\mathbf{I}_{f_i} \in \mathbb{R}^{3\times 3}$ is a constant symmetric positive definite inertia matrix of the i^{th} rigid body with respect to $\mathcal{F}_i$, and the vector $\boldsymbol{\Gamma}_i$ is the external torque input expressed in $\mathcal{F}_i$. The unit-quaternion $\mathbf{Q}_i = (\mathbf{q}_i^\top, \eta_i)^\top$ is composed of a vector part $\mathbf{q}_i \in \mathbb{R}^3$ and a scalar part $\eta_i \in \mathbb{R}$, and represents the orientation of the i^{th} rigid body. The elements of the unit-quaternion $\mathbf{Q}_i$ satisfy the unity constraint: $\eta_i^2 + \mathbf{q}_i^\top \mathbf{q}_i = 1$. The matrix $\mathbf{T}(\mathbf{Q}_i) \in \mathbb{R}^{4\times 3}$ is given by

$$\mathbf{T}(\mathbf{Q}_i) = \begin{pmatrix} \eta_i \, \mathbf{I}_3 + \mathbf{S}(\mathbf{q}_i) \\ -\mathbf{q}_i^\top \end{pmatrix} \tag{2}$$

and satisfies: $\mathbf{T}(\mathbf{Q}_i)^\top \mathbf{T}(\mathbf{Q}_i) = \mathbf{I}_3$, where $\mathbf{I}_3$ is the 3-by-3 identity matrix, and $\mathbf{S}(\mathbf{x})$ is the skew-symmetric matrix such that $\mathbf{S}(\mathbf{x}_1)\mathbf{x}_2 = \mathbf{x}_1 \times \mathbf{x}_2$ for any vectors $\mathbf{x}_1 \in \mathbb{R}^3$ and $\mathbf{x}_2 \in \mathbb{R}^3$, where '$\times$' denotes the vector cross product. The orthogonal rotation matrix $\mathbf{R}(\mathbf{Q}_i) \in SO(3)$ related to the unit-quaternion $\mathbf{Q}_i$, that brings the inertial frame into the i^{th} body frame, can be obtained through the Rodriguez formula as: $\mathbf{R}(\mathbf{Q}_i) = (\eta_i^2 - \mathbf{q}_i^\top \mathbf{q}_i)\mathbf{I}_3 + 2\mathbf{q}_i\mathbf{q}_i^\top - 2\eta_i \mathbf{S}(\mathbf{q}_i)$. The time-derivative of the rotation matrix $\mathbf{R}(\mathbf{Q}_i)$ is given as: $\dot{\mathbf{R}}(\mathbf{Q}_i) = -\mathbf{S}(\boldsymbol{\omega}_i)\mathbf{R}(\mathbf{Q}_i)$. The multiplication between two unit-quaternion, $\mathbf{Q}_i = (\mathbf{q}_i^\top, \eta_i)^\top$ and $\mathbf{Q}_j = (\mathbf{q}_j^\top, \eta_j)^\top$, is defined by the following operation: $\mathbf{Q}_i \odot \mathbf{Q}_j = ((\eta_i \mathbf{q}_j + \eta_j \mathbf{q}_i + \mathbf{S}(\mathbf{q}_i)\mathbf{q}_j)^\top, \eta_i\eta_j - \mathbf{q}_i^\top \mathbf{q}_j)^\top$. The inverse or conjugate of the unit-quaternion $\mathbf{Q}_i$ is defined by, $\mathbf{Q}_i^{-1} = (-\mathbf{q}_i^\top, \eta_i)^\top$, with the quaternion identity given by $\mathbf{Q}_I := (\mathbf{0}_3^\top, 1)^\top$, where $\mathbf{0}_m \in \mathbb{R}^m$ is the vector of zero elements. Note that due to the redundancy in the unit-quaternion representation, $\pm\mathbf{Q}_I$ represents the same physical orientation. For more properties of the unit-quaternion representation of the attitude, the reader is referred to [21].

B. Problem Statement

To achieve attitude synchronization, rigid bodies in the team must exchange some of their states information. We assume that the information flow between members of the team is fixed and is represented by a weighted graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{K})$, where $\mathcal{N}$ is the set of nodes, or vertices, describing the set of vehicles in the team, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of pairs of nodes, called edges, and $\mathcal{K} = [k_{ij}]$ is a weighted adjacency matrix. An edge $(i,j) \in \mathcal{E}$ indicates that the i^{th} rigid body receives information from the j^{th} rigid body, which is designated as its neighbor. The weighted adjacency matrix of a weighted graph is defined such that $k_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$ and $k_{ij} = 0$ if and only if $(i,j) \notin \mathcal{E}$. If the interconnection between rigid bodies is bidirectional, then $\mathcal{G}$ is undirected, the pairs of nodes in $\mathcal{E}$ are unordered, $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$, and $\mathcal{K}$ is symmetric, i.e., $k_{ij} = k_{ji}$. In the case of unidirectional communication topology, $\mathcal{G}$ is a directed graph, $\mathcal{E}$ contains ordered pairs, and $\mathcal{K}$ is not necessarily symmetric. An undirected graph is said to be connected if there is an undirected path between any two distinct nodes of the graph. Similarly, a directed graph is said to be strongly connected if there exists a directed path between any two distinct nodes [22]. We also assume that each rigid body can sense its states with no delays, and the communication between the i^{th} and j^{th} rigid bodies, with $(i,j) \in \mathcal{E}$, is delayed by τ_{ij}, with τ_{ij} not necessarily equal to τ_{ji}. With the above assumptions, our objective is to design control laws for each rigid body without angular velocity measurements in the presence of communication delays, such that the following problems are solved:

- Leaderless synchronization problem $\mathbf{LSP}$. When no desired attitude is assigned to the team, all rigid bodies are required to synchronize their attitudes to the same attitude, such that $\boldsymbol{\omega}_i \to \mathbf{0}_3$, and $\mathbf{Q}_i \to \mathbf{Q}_j$, for all $i, j \in \mathcal{N}$.
- Leader-follower problem $\mathbf{LFP}$. Given a constant desired attitude, represented by the unit-quaternion $\mathbf{Q}_d := (\mathbf{q}_d^\top, \eta_d)^\top$, available to a single rigid body in the team acting as a leader. All rigid bodies are required to synchronize their attitudes to the desired attitude, i.e., $\boldsymbol{\omega}_i \to \mathbf{0}_3$ and $\mathbf{Q}_i \to \mathbf{Q}_d$, for all $i \in \mathcal{N}$.

C. Virtual Systems-Based Approach

In this subsection, we present an approach to the attitude synchronization problem that can handle communication delays and removes the requirement of angular velocity measurements. This approach is based on *virtual* systems. Let us associate to each rigid body the following virtual system

$$\dot{\mathbf{Q}}_{v_i} = \frac{1}{2}\mathbf{T}\left(\mathbf{Q}_{v_i}\right)\boldsymbol{\omega}_{v_i} \tag{3}$$

for $i \in \mathcal{N}$, where $\mathbf{Q}_{v_i} = (\mathbf{q}_{v_i}^\top, \eta_{v_i})^\top$ is the unit-quaternion representing the attitude of the virtual system (3), with $\mathbf{Q}_{v_i}(0)$ can be initialized arbitrarily, and $\boldsymbol{\omega}_{v_i}$ is the virtual angular velocity input of the virtual system, which will be designed later. The matrix $\mathbf{T}(\mathbf{Q}_{v_i})$ can be obtained similar to (2) as: $\mathbf{T}(\mathbf{Q}_{v_i}) = \begin{pmatrix} \eta_{v_i}\mathbf{I}_3 + \mathbf{S}(\mathbf{q}_{v_i}) \\ -\mathbf{q}_{v_i}^\top \end{pmatrix}$. We let the discrepancy between the attitude of the i^{th} rigid body and its corresponding virtual system be represented by the unit-quaternion $\mathbf{Q}_i^e := (\mathbf{q}_i^{e\top}, \eta_i^e)^\top$, which is defined as: $\mathbf{Q}_i^e = \mathbf{Q}_{v_i}^{-1} \odot \mathbf{Q}_i$, and satisfies the unit-quaternion dynamics

$$\dot{\mathbf{Q}}_i^e = \frac{1}{2}\mathbf{T}\left(\mathbf{Q}_i^e\right)\boldsymbol{\omega}_i^e, \quad \boldsymbol{\omega}_i^e = \boldsymbol{\omega}_i - \mathbf{R}\left(\mathbf{Q}_i^e\right)\boldsymbol{\omega}_{v_i} \tag{4}$$

where $\mathbf{T}(\mathbf{Q}_i^e)$ can be obtained similar to (2), and $\mathbf{R}(\mathbf{Q}_i^e)$ is the rotational matrix related to $\mathbf{Q}_i^e$ and is given as $\mathbf{R}(\mathbf{Q}_i^e) = \mathbf{R}(\mathbf{Q}_i)\mathbf{R}(\mathbf{Q}_{v_i})^\top$, [21]. The main idea in this approach is to design the input torque of each rigid body, $\boldsymbol{\Gamma}_i$, without angular velocity measurements such that the states of each rigid body converge asymptotically to the states of its corresponding virtual system. Then, our control objectives will be achieved if one determines an appropriate input of each virtual system (3) such that all virtual systems synchronize their attitudes in the presence of communication delays.

III. Attitude Synchronization Without Angular Velocity Measurements

First, we consider the design of the input torque for each rigid body. Based on the auxiliary systems approach introduced in [23], we propose the following input torque in (1)

$$\boldsymbol{\Gamma}_i = \mathbf{H}_i\left(\dot{\boldsymbol{\omega}}_{v_i}, \boldsymbol{\omega}_{v_i}, \mathbf{Q}_i^e\right) - k_i^p \mathbf{q}_i^e - k_i^d \tilde{\mathbf{q}}_i^e \tag{5}$$

for $i \in \mathcal{N}$, where $\mathbf{H}_i(\dot{\boldsymbol{\omega}}_{v_i}, \boldsymbol{\omega}_{v_i}, \mathbf{Q}_i^e) = (\mathbf{I}_{f_i}\mathbf{R}(\mathbf{Q}_i^e)\dot{\boldsymbol{\omega}}_{v_i} + \mathbf{S}(\mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i})\mathbf{I}_{f_i}\mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i})$, k_i^p and k_i^d are strictly positive scalar gains, $\mathbf{q}_i^e$ is the vector part of the unit-quaternion $\mathbf{Q}_i^e$ defined in (4), and $\tilde{\mathbf{q}}_i^e$ is the vector part of the unit-quaternion $\tilde{\mathbf{Q}}_i^e := (\tilde{\mathbf{q}}_i^{e\top}, \tilde{\eta}_i^e)^\top$ defined as

$$\tilde{\mathbf{Q}}_i^e = \mathbf{P}_i^{-1} \odot \mathbf{Q}_i^e, \quad \dot{\mathbf{P}}_i = \frac{1}{2}\mathbf{T}(\mathbf{P}_i)\boldsymbol{\beta}_i \tag{6}$$

where $\mathbf{P}_i$ is a unit-quaternion that can be initialized arbitrarily, $\mathbf{T}(\mathbf{P}_i)$ is given similar to (2), and $\boldsymbol{\beta}_i \in \mathbb{R}^3$ is an input to be determined. We can verify that $\tilde{\mathbf{Q}}_i^e$ satisfies the following unit-quaternion dynamics

$$\dot{\tilde{\mathbf{Q}}}_i^e = \frac{1}{2}\mathbf{T}(\tilde{\mathbf{Q}}_i^e)\tilde{\boldsymbol{\omega}}_i^e, \quad \tilde{\boldsymbol{\omega}}_i^e = \boldsymbol{\omega}_i^e - \mathbf{R}(\tilde{\mathbf{Q}}_i^e)\boldsymbol{\beta}_i \tag{7}$$

with $\mathbf{T}(\tilde{\mathbf{Q}}_i^e)$ being defined similar to (2). The aim from the above design of the input torque is to drive the states of each rigid body to the states of its corresponding virtual system, i.e., $\boldsymbol{\omega}_i \to \boldsymbol{\omega}_{v_i}$ and $\mathbf{Q}_i \to \mathbf{Q}_{v_i}$, and equivalently $\boldsymbol{\omega}_i^e \to \mathbf{0}_3$ and $\mathbf{Q}_i^e \to \pm\mathbf{Q}_I$ for $i \in \mathcal{N}$. It should be noted that the torque input (5) is given in terms of the virtual states, i.e., $\mathbf{Q}_{v_i}, \boldsymbol{\omega}_{v_i}$ and $\dot{\boldsymbol{\omega}}_{v_i}$, and the absolute attitudes of the rigid bodies, and does not depend on the angular velocity of the rigid bodies. In view of this, the remaining part of the design is to determine an appropriate virtual angular velocity and its time derivative, i.e., $\boldsymbol{\omega}_{v_i}$ and $\dot{\boldsymbol{\omega}}_{v_i}$, such that our control objectives are achieved in the presence of communication delays.

A. *Control Scheme I*

Consider the case where the interconnection graph between neighboring rigid bodies is bidirectional, represented by the undirected graph $\mathcal{G}$, and is subject to time-varying communication delays. To achieve attitude synchronization based on the virtual systems introduced above, communicating rigid bodies must transmit the attitudes of their corresponding virtual systems. Since the communication between rigid bodies is delayed, we let the relative attitude between the i^{th} and j^{th} virtual systems be represented by the unit-quaternion $\bar{\mathbf{Q}}_{v_{ij}} := (\bar{\mathbf{q}}_{v_{ij}}^\top, \bar{\eta}_{v_{ij}})^\top$, defined as

$$\bar{\mathbf{Q}}_{v_{ij}} = \mathbf{Q}_{v_j}^{-1}(t - \tau_{ij}) \odot \mathbf{Q}_{v_i}. \tag{8}$$

In addition, in the case where the constant desired attitude, represented by $\mathbf{Q}_d$, is available to a single rigid body in the team, case of the $\mathbf{LFP}$, the attitude error between the desired attitude and the virtual system associated to the leader rigid body is represented by the unit-quaternion $\tilde{\mathbf{Q}}_{v_l} := (\tilde{\mathbf{q}}_{v_l}^\top, \tilde{\eta}_{v_l})^\top$, defined as: $\tilde{\mathbf{Q}}_{v_l} = \mathbf{Q}_d^{-1} \odot \mathbf{Q}_{v_l}$, satisfying the unit-quaternion dynamics

$$\dot{\tilde{\mathbf{Q}}}_{v_l} = \frac{1}{2}\mathbf{T}(\tilde{\mathbf{Q}}_{v_l})\boldsymbol{\omega}_{v_l} \tag{9}$$

with $\mathbf{T}(\tilde{\mathbf{Q}}_{v_l})$ being defined similar to (2), and the subscript "l" is used to designate the leader. With the above definitions, we propose the following design of the virtual angular velocity:

$$\dot{\boldsymbol{\omega}}_{v_i} = -k_i^\omega \boldsymbol{\omega}_{v_i} - \alpha\bar{\mathbf{u}}_i - \sum_{j=1}^n k_{ij}\bar{\mathbf{q}}_{v_{ij}} \tag{10}$$

for $i \in \mathcal{N}$, where $\boldsymbol{\omega}_{v_i}(0)$ can be selected arbitrarily, $\bar{\mathbf{q}}_{v_{ij}}$ is the vector part of the unit-quaternion $\bar{\mathbf{Q}}_{v_{ij}}$ defined in (8), $\bar{\mathbf{u}}_i = k_l^q\tilde{\mathbf{q}}_{v_l}$, for $i = l$, and $\bar{\mathbf{u}}_i = 0$, for $i \neq l$, $\tilde{\mathbf{q}}_{v_l}$ is the vector part of the unit-quaternion $\tilde{\mathbf{Q}}_{v_l}$ defined in (9), with the subscript "l" being used to designate the leader. The scalar gains $k_l^q > 0$, $k_i^\omega > 0$ for $i \in \mathcal{N}$, and $k_{ij} \geq 0$ is the $(i,j)^{th}$ entry of the adjacency matrix of the weighted undirected graph $\mathcal{G}$. The scalar α is selected as: $\alpha = 0$ for the $\mathbf{LSP}$ and $\alpha = 1$ for the $\mathbf{LFP}$. Under the assumption that neighboring rigid bodies can communicate the attitude of their corresponding virtual systems, i.e., $\mathbf{Q}_{v_i}$, the following result holds:

Theorem 1: Consider system (1) with the torque input law (5) with (3) and (10). Let the time-varying communication delays be bounded such that $\tau_{ij} \leq \tau$ for $(i,j) \in \mathcal{E}$, where τ is a positive constant, and let the controller gains satisfy: $k_i^z := k_i^\omega - \sum_{j=1}^n (k_{ij}/4)(\epsilon + (\tau^2/\epsilon)) > 0$, for some $\epsilon > 0$. Let the vector $\boldsymbol{\beta}_i$ in (6) be given as: $\boldsymbol{\beta}_i = \lambda_i\tilde{\mathbf{q}}_i^e$, with λ_i a strictly positive scalar gain. If the undirected communication graph is a tree[1], then all the signals are globally bounded and the $\mathbf{LSP}$ and the $\mathbf{LFP}$ are solved by setting $\alpha = 0$, 1 respectively. Moreover, if there exists a time $t_0 > 0$ such that $\eta_{v_i}(t) > 0$ (or $\eta_{v_i}(t) < 0$) for $t \geq t_0$, then the above results hold for any connected undirected communication graph.

Proof: See Appendix A.

Remark 1: Note that the above attitude synchronization scheme can be extended to the case where the angular velocities are available for feedback. In this case, the virtual systems are not required and we can show, following similar steps as in the proof of Theorem 1, that the $\mathbf{LSP}$ and the $\mathbf{LFP}$ will be solved under the same conditions reported in Theorem 1 if the following control input is implemented: $\boldsymbol{\Gamma}_i = -\alpha\bar{\mathbf{u}}_i - k_i^\omega\boldsymbol{\omega}_i - \sum_{i=1}^n k_{ij}\bar{\mathbf{q}}_{ij}$, where the control gains are defined as in Theorem 1, $\bar{\mathbf{q}}_{ij}$ is the vector part of the unit-quaternion $\bar{\mathbf{Q}}_{ij} = \mathbf{Q}_j^{-1}(t - \tau_{ij}) \odot \mathbf{Q}_i$, and $\bar{\mathbf{u}}_i = k_l^q\tilde{\mathbf{q}}_l$, for $i = l$, and $\bar{\mathbf{u}}_i = 0$, for $i \neq l$, with $\tilde{\mathbf{q}}_l$ being the vector part of the unit-quaternion $\tilde{\mathbf{Q}}_l = \mathbf{Q}_d^{-1} \odot \mathbf{Q}_l$.

B. *Control Scheme II*

Note that in the above control scheme the communication graph is restricted to be an undirected tree. It is clear that this restriction is due to the nonlinear expression of the relative attitudes in (8) and the proposed design of the virtual angular velocity in (10). To relax the condition on the communication graph, we present in this subsection a different design of the virtual angular velocity for each rigid body.

We assume that the communication delays are constant and the interconnection between rigid bodies is unidirectional and is represented by the directed graph $\mathcal{G}$. To solve the $\mathbf{LSP}$ in this case, we propose the following design of the virtual angular velocity in (3) and (5)

$$\boldsymbol{\omega}_{v_i} = -\sum_{j=1}^n k_{ij}\left(\mathbf{q}_{v_i} - \mathbf{q}_{v_j}(t - \tau_{ij})\right) \tag{11}$$

for $i \in \mathcal{N}$, where $k_{ij} \geq 0$ is the $(i,j)^{th}$ entry of the adjacency matrix of the directed communication graph $\mathcal{G}$. It is easy to verify that

$$\dot{\boldsymbol{\omega}}_{v_i} = -\sum_{j=1}^n k_{ij}\left(\dot{\mathbf{q}}_{v_i} - \dot{\mathbf{q}}_{v_j}(t - \tau_{ij})\right) \tag{12}$$

with $\dot{\mathbf{q}}_{v_i} = (1/2)(\eta_{v_i}\mathbf{I}_3 + \mathbf{S}(\mathbf{q}_{v_i}))\boldsymbol{\omega}_{v_i}$. Under the assumption that neighboring rigid bodies can communicate the states of their corresponding virtual systems, i.e., $\mathbf{Q}_{v_i}$ and $\boldsymbol{\omega}_{v_i}$, the following result holds:

Theorem 2: Consider system (1) with the control law (5) with (3) and (11), (12). Let the vector $\boldsymbol{\beta}_i$ in (6) be given as in Theorem 1. If the directed communication graph is strongly connected, then all the signals are globally bounded and the $\mathbf{LSP}$ is solved in the presence of arbitrary constant communication delays.

Proof: See Appendix B.

Remark 2: It should be noted that the above control scheme can be extended to the full state information case in a similar manner described in Remark 1. In addition, the input torque (5) with (11) and (12) consists of pure unit-quaternion terms and the inertia matrix of the rigid

[1]An undirected graph is a tree if it is connected and acyclic, [22].

body. As a result, a natural saturation is achieved for the control effort as follows: $\|\mathbf{\Gamma}_i\| \leq \|\mathbf{I}_{f_i}\|(\varrho_i+\rho_i^2)+k_i^p+k_i^d$, with ρ_i and ϱ_i can be obtained respectively from (11) and (12) as: $\|\boldsymbol{\omega}_{v_i}\| \leq \rho_i := 2\sum_{j=1}^n k_{ij}$, and $\|\dot{\boldsymbol{\omega}}_{v_i}\| \leq \varrho_i := (1/2)\rho_i^2$.

C. Extension

We show in the following that the attitude synchronization scheme in Theorem 2 can be modified to solve the cooperative attitude tracking control problem ($\mathbf{CATP}$). In this case, a time-varying desired trajectory, $(\mathbf{Q}_d, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d)$, is available to all rigid bodies in the team, and it is required that all rigid bodies synchronize their attitudes to the desired attitude, while minimizing the relative attitudes between rigid bodies during the transient in the presence of communication delays, i.e., $\mathbf{Q}_i \to \mathbf{Q}_d$ and $\boldsymbol{\omega}_i \to \boldsymbol{\omega}_d$. To solve this problem, we define the attitude tracking error between each virtual system and the desired attitude as: $\tilde{\mathbf{Q}}_{v_i} := (\tilde{\mathbf{q}}_{v_i}^\top, \tilde{\eta}_{v_i})^\top = \mathbf{Q}_d^{-1} \odot \mathbf{Q}_{v_i}$, which is governed by the dynamics $\dot{\tilde{\mathbf{Q}}}_{v_i} = (1/2)\mathbf{T}(\tilde{\mathbf{Q}}_{v_i})\tilde{\boldsymbol{\omega}}_{v_i}$, with $\tilde{\boldsymbol{\omega}}_{v_i} = \boldsymbol{\omega}_{v_i} - \mathbf{R}(\tilde{\mathbf{Q}}_{v_i})\boldsymbol{\omega}_d$ and $\mathbf{T}(\tilde{\mathbf{Q}}_{v_i})$ can be obtained similar to (2). With these definitions, we propose the following virtual angular velocity in (1) and (5) in the case of constant communication delays

$$\boldsymbol{\omega}_{v_i} = \mathbf{R}(\tilde{\mathbf{Q}}_{v_i})\boldsymbol{\omega}_d - k_i^q\tilde{\mathbf{q}}_{v_i} - \sum_{j=1}^n k_{ij}\left(\tilde{\mathbf{q}}_{v_i} - \tilde{\mathbf{q}}_{v_j}(t-\tau_{ij})\right) \quad (13)$$

where $k_i^q > 0$ and k_{ij} are defined as in Theorem 2. The time derivative of $\dot{\boldsymbol{\omega}}_{v_i}$ can be explicitly computed as

$$\dot{\boldsymbol{\omega}}_{v_i} = \frac{d}{dt}\left(\mathbf{R}(\tilde{\mathbf{Q}}_{v_i})\boldsymbol{\omega}_d\right) - k_i^q\dot{\tilde{\mathbf{q}}}_{v_i} - \sum_{j=1}^n k_{ij}\left(\dot{\tilde{\mathbf{q}}}_{v_i} - \dot{\tilde{\mathbf{q}}}_{v_j}(t-\tau_{ij})\right) \quad (14)$$

with $\dot{\mathbf{R}}(\tilde{\mathbf{Q}}_{v_i}) = -\mathbf{S}(\tilde{\boldsymbol{\omega}}_{v_i})\mathbf{R}(\tilde{\mathbf{Q}}_{v_i})$, and $\dot{\tilde{\mathbf{Q}}}_{v_i}$ is given above. Under the assumption that $\boldsymbol{\omega}_d$ and $\dot{\boldsymbol{\omega}}_d$ are bounded, and neighboring rigid bodies can communicate the states of their corresponding virtual systems, i.e., $\tilde{\mathbf{Q}}_{v_i}$ and $\tilde{\boldsymbol{\omega}}_{v_i}$, the following result holds:

Theorem 3: Consider system (1) with the control law (5), (3) and (13), (14). Let the vector $\boldsymbol{\beta}_i$ in (6) be given as in Theorem 1. If the directed communication graph is strongly connected, then all the signals are globally bounded and the $\mathbf{CATP}$ is solved in the presence of arbitrary constant communication delays.

Proof: The proof follows similar steps as the proof of Theorem 2 and is omitted due to space limitations.

Remark 3: The control scheme in Theorem 3 can be *a priori* bounded as: $\|\mathbf{\Gamma}_i\| \leq \|\mathbf{I}_{f_i}\|(\bar{\varrho}_i + \bar{\rho}_i^2) + k_i^p + k_i^d$, with $\bar{\rho}_i$ and $\bar{\varrho}_i$ are obtained respectively as: $\|\boldsymbol{\omega}_{v_i}\| \leq \bar{\rho}_i := \|\boldsymbol{\omega}_d\| + \kappa_i$, $\|\dot{\boldsymbol{\omega}}_{v_i}\| \leq \bar{\varrho}_i := \|\dot{\boldsymbol{\omega}}_d\| + \|\boldsymbol{\omega}_d\|\kappa_i + (1/2)\kappa_i^2$, with $\kappa_i = k_i^q + 2\sum_{j=1}^n k_{ij}$. Also, the extension of this control law to the full state information case follows the same lines in Remark 1.

Remark 4: It is worth pointing out that the attitude synchronization scheme in Theorem 1 can also be extended to solve the $\mathbf{CATP}$ in the case of undirected communication topologies subject to time-varying communication delays. In this case, the relative attitude between virtual systems, given in (8), should be redefined as: $\bar{\mathbf{Q}}_{v_{ij}} = \tilde{\mathbf{Q}}_{v_j}^{-1}(t-\tau_{ij}) \odot \tilde{\mathbf{Q}}_{v_i}$, and the virtual angular velocity, given in (10), should be modified as: $\dot{\boldsymbol{\omega}}_{v_i} = -k_i^q\tilde{\mathbf{q}}_{v_i} - k_i^\omega\tilde{\boldsymbol{\omega}}_{v_i} - \sum_{j=1}^n k_{ij}\bar{\mathbf{q}}_{v_{ij}} + \mathbf{R}(\tilde{\mathbf{Q}}_{v_i})\dot{\boldsymbol{\omega}}_d - \mathbf{S}(\tilde{\boldsymbol{\omega}}_{v_i})\mathbf{R}(\tilde{\mathbf{Q}}_{v_i})\boldsymbol{\omega}_d$, with the control gains defined as in Theorem 1, $k_i^q > 0$, and $\tilde{\mathbf{q}}_{v_i}$ and $\tilde{\boldsymbol{\omega}}_{v_i}$ are defined as in Theorem 3. Following similar steps as in the proof of Theorem 1, with the assumption that $\boldsymbol{\omega}_d$ and $\dot{\boldsymbol{\omega}}_d$ are bounded, we can show that the $\mathbf{CATP}$ will be solved under any undirected communication graph with similar conditions on the control gains and the delays upper bound given in Theorem 1.

IV. Conclusion

We addressed the attitude synchronization problem for a group of rigid bodies without angular velocity measurements and in the presence of communication delays. To solve this problem, we proposed a virtual systems-based approach that reduces the problem to a separate design of a tracking control law, without angular velocity measurements, and a synchronization algorithm with communication delays using the internally synthesized virtual states. To the best of our knowledge, the synchronization problem in the presence of communication delays has never been considered in the partial state information case, even for multi-agent systems modeled as linear double integrators.

Based on this approach, we proposed in Theorem 1 a solution to the $\mathbf{LSP}$ and the $\mathbf{LFP}$ problems in the presence of time-varying communication delays. We have shown that attitude synchronization is achieved under a sufficient condition that can be satisfied with an appropriate choice of the control gains and the reasonable assumption that the upper bound of the communication delays is known. Also, the undirected communication graph is restricted to be a tree. With a different design of the virtual systems inputs, Theorem 2 presents a solution to the $\mathbf{LSP}$ under a strongly connected directed communication graph in the presence of arbitrary constant communication delays. This removes the restrictions obtained in Theorem 1 in this case, and considers a more general communication topology between rigid bodies. The extension of this result to the case of the $\mathbf{LFP}$, as well as the case of time-varying communication delays is not straightforward and will be examined in our future work.

As mentioned earlier, very few papers have considered the attitude synchronization problem with delayed communication in the full state information case. The authors in [20] have addressed the $\mathbf{LSP}$ with constant communication delays in the full state information case. In this work, only the attitude kinematics have been considered to design a desired angular velocity that achieves attitude synchronization under strongly connected directed graphs. In addition, the result of this paper relies on the assumption that the rotation matrix of each rigid body is always positive definite. Besides the non requirement of angular velocity measurements, the result of Theorem 2 provides an input torque design that solves this problem with no assumptions on the attitudes of the rigid bodies in the team.

In [16]–[19], the attitude dynamics have been considered to design attitude synchronization schemes with delayed communication. In these papers, only the $\mathbf{CATP}$ under undirected communication topologies have been addressed. The definition of the error variables in these works and the Lyapunov-Krasovskii functionals used in the analysis make their extension to solve the $\mathbf{LSP}$ and the $\mathbf{LFP}$ in the full state information case not trivial. Theorem 1 provides solutions to these problems and removes the requirements of angular velocity measurements. In addition, we have shown that our approach can be modified to solve the $\mathbf{CATP}$ in the presence of constant (time-varying) communication delays under directed (undirected) communication graphs. Furthermore, the proposed attitude synchronization schemes can be extended in a straightforward manner to the full state information case.

Moreover, the result of Theorem 1 carries an additional feature, which consists in the fact that the time-varying communication delays are only assumed to be bounded. Also, the control schemes in Theorem 2 and Theorem 3 are guaranteed to be *a priori* bounded. This enables the designer to select appropriate control gains to account for input saturations. The extension of this work to the case of dynamically switching topologies is a challenging problem and will be the focus of our future work.

APPENDIX A
PROOF OF THEOREM 1

To prepare the proof of the theorem, we derive first the closed loop dynamics of the system and define some variables and results required in the proof. Consider the time-derivative of the angular velocity error $\boldsymbol{\omega}_i^e$, which can be written from (4) as: $\dot{\boldsymbol{\omega}}_i^e = \dot{\boldsymbol{\omega}}_i - \mathbf{R}(\mathbf{Q}_i^e)\dot{\boldsymbol{\omega}}_{v_i} + \mathbf{S}(\boldsymbol{\omega}_i^e)\mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i}$. Using the attitude dynamics (1) and (4), and after some algebraic manipulations, we can write

$$\begin{aligned}\mathbf{I}_{f_i}\dot{\boldsymbol{\omega}}_i^e =& \boldsymbol{\Gamma}_i - \mathbf{S}(\boldsymbol{\omega}_i^e)\mathbf{I}_{f_i}(\boldsymbol{\omega}_i^e + \mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i}) - \mathbf{H}_i(\dot{\boldsymbol{\omega}}_{v_i}, \boldsymbol{\omega}_{v_i}, \mathbf{Q}_i^e) \\ &- (\mathbf{I}_{f_i}\mathbf{S}(\mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i}) + \mathbf{S}(\mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i})\mathbf{I}_{f_i})\boldsymbol{\omega}_i^e \end{aligned} \quad (15)$$

with $\mathbf{H}_i(\dot{\boldsymbol{\omega}}_{v_i}, \boldsymbol{\omega}_{v_i}, \mathbf{Q}_i^e)$ is defined after (5). Since $\mathbf{I}_{f_i} = \mathbf{I}_{f_i}^\top > 0$, we can verify that $(\mathbf{I}_{f_i}\mathbf{S}(\mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i}) + \mathbf{S}(\mathbf{R}(\mathbf{Q}_i^e)\boldsymbol{\omega}_{v_i})\mathbf{I}_{f_i})$ is skew symmetric, and therefore, we obtain: ${\boldsymbol{\omega}_i^e}^\top \mathbf{I}_{f_i}\dot{\boldsymbol{\omega}}_i^e = {\boldsymbol{\omega}_i^e}^\top(\boldsymbol{\Gamma}_i - \mathbf{H}_i(\dot{\boldsymbol{\omega}}_{v_i}, \boldsymbol{\omega}_{v_i}, \mathbf{Q}_i^e))$.

Also, for analysis purposes, we let the discrepancy between the attitudes of the i^{th} and j^{th} virtual systems, in the case of no communication delays, be represented by the unit-quaternion

$$\mathbf{Q}_{v_{ij}} := \left(\mathbf{q}_{v_{ij}}^\top, \eta_{v_{ij}}\right)^\top = \mathbf{Q}_{v_j}^{-1} \odot \mathbf{Q}_{v_i} \quad (16)$$

which satisfies the unit-quaternion dynamics: $\dot{\mathbf{Q}}_{v_{ij}} = (1/2)\mathbf{T}(\mathbf{Q}_{v_{ij}})\boldsymbol{\omega}_{v_{ij}}$, with $\boldsymbol{\omega}_{v_{ij}} = \boldsymbol{\omega}_{v_i} - \mathbf{R}(\mathbf{Q}_{v_{ij}})\boldsymbol{\omega}_{v_j}$, $\mathbf{T}(\mathbf{Q}_{v_{ij}})(\mathbf{Q}_{v_{ij}})$ can be obtained similar to (2), and $\mathbf{R}(\mathbf{Q}_{v_{ij}}) = \mathbf{R}(\mathbf{Q}_{v_i})\mathbf{R}(\mathbf{Q}_{v_j})^\top$. It should be noted that $\mathbf{Q}_{v_{ij}}$ and $\bar{\mathbf{Q}}_{v_{ij}}$, given respectively in (16) and (8), represent two different attitude errors. However, by exploiting the definition of the unit-quaternion multiplication and the attitude dynamics (1), we can show the result of the following lemma, which is proved in Appendix C.

Lemma 1: Consider the relative attitudes defined in (16) and (8). Then, the following relation holds for any strictly positive constant ϵ:

$$(\bar{\mathbf{q}}_{v_{ij}} - \mathbf{q}_{v_{ij}})^\top \boldsymbol{\omega}_{v_i} \leq \epsilon \dot{\mathbf{Q}}_{v_i}^\top \dot{\mathbf{Q}}_{v_i} + \frac{\tau_{ij}}{\epsilon}\int_{t-\tau_{ij}}^{t} \dot{\mathbf{Q}}_{v_j}^\top \dot{\mathbf{Q}}_{v_j} ds.$$

Now, consider the Lyapunov-Krasovskii functional candidate; $V = V_1 + V_2$, with

$$\begin{aligned} V_1 =& \sum_{i=1}^n \left(\frac{1}{2}{\boldsymbol{\omega}_i^e}^\top \mathbf{I}_{f_i}\boldsymbol{\omega}_i^e + 2k_i^p(1-\eta_i^e) + 2k_i^d(1-\tilde{\eta}_i^e)\right), \\ V_2 =& 2\alpha k_l^q(1-\tilde{\eta}_{v_l}) + \sum_{i=1}^n\left(\frac{1}{2}\boldsymbol{\omega}_{v_i}^\top\boldsymbol{\omega}_{v_i} + \sum_{j=1}^n k_{ij}\left(1-\eta_{v_{ij}}\right)\right) \\ &+ \sum_{i=1}^n\sum_{j=1}^n \frac{k_{ij}\tau}{\epsilon}\int_{-\tau}^{0}\int_{t+s}^{t}\dot{\mathbf{Q}}_{v_j}^\top(\sigma)\dot{\mathbf{Q}}_{v_j}(\sigma)d\sigma ds \end{aligned}$$

where $\epsilon > 0, \tau_{ij} \leq \tau$, with τ being a positive constant, $\tilde{\eta}_{v_l}$ is the scalar part of $\tilde{\mathbf{Q}}_{v_l}$ defined in (9), $\eta_{v_{ij}}$ is the scalar part of $\mathbf{Q}_{v_{ij}}$ defined in (16), η_i^e is the scalar part of $\mathbf{Q}_i^e$ defined in (4), and $\tilde{\eta}_i^e$ is the scalar part of $\tilde{\mathbf{Q}}_i^e$ defined in (6). Note that $2(1-\eta_{v_{ij}}) = (\mathbf{q}_{v_{ij}}^T\mathbf{q}_{v_{ij}} + (1-\eta_{v_{ij}})^2)$, and similar relations hold for the elements of $\tilde{\mathbf{Q}}_{v_l}$, $\mathbf{Q}_i^e$ and $\tilde{\mathbf{Q}}_i^e$. The time derivative of V_1 evaluated along the dynamics (15), using (4), (5) and (7), is obtained as

$$\dot{V}_1 = -\sum_{i=1}^n k_i^d\lambda_i{\tilde{\mathbf{q}}_i^e}^\top\tilde{\mathbf{q}}_i^e \quad (17)$$

where we have used the expression of $\boldsymbol{\beta}_i$ given in the theorem and the relation $\mathbf{R}(\tilde{\mathbf{Q}}_i^e)\tilde{\mathbf{q}}_i^e = \tilde{\mathbf{q}}_i^e$. The time derivative of V_2 evaluated along (10) gives

$$\begin{aligned}\dot{V}_2 =& \alpha k_l^q\tilde{\mathbf{q}}_{v_l}^\top\boldsymbol{\omega}_{v_l} + \sum_{i=1}^n\boldsymbol{\omega}_{v_i}^\top\left(-\alpha\bar{\mathbf{u}}_i - k_i^\omega\boldsymbol{\omega}_{v_i} - \sum_{j=1}^n k_{ij}\bar{\mathbf{q}}_{v_{ij}}\right) \\ &+ \sum_{i=1}^n\sum_{j=1}^n k_{ij}\left(\frac{1}{2}\mathbf{q}_{v_{ij}}^\top\boldsymbol{\omega}_{v_{ij}} + \frac{\tau}{\epsilon}\left(\tau\dot{\mathbf{Q}}_{v_j}^\top\dot{\mathbf{Q}}_{v_j} - \int_{t-\tau}^{t}\dot{\mathbf{Q}}_{v_j}^\top\dot{\mathbf{Q}}_{v_j}ds\right)\right).\end{aligned}$$

Since the communication graph is undirected, we know that $k_{ij} = k_{ji}$, and using the definition of $\boldsymbol{\omega}_{v_{ij}}$ we can show that, [6]: $1/2\sum_{i=1}^n\sum_{j=1}^n k_{ij}\boldsymbol{\omega}_{v_{ij}}^\top\mathbf{q}_{v_{ij}} = \sum_{i=1}^n\sum_{j=1}^n k_{ij}\boldsymbol{\omega}_{v_i}^\top\mathbf{q}_{v_{ij}}$. Using the result in Lemma 1, and the relations: $\sum_{i=1}^n\boldsymbol{\omega}_{v_i}^\top(-\alpha\bar{\mathbf{u}}_i) = -\alpha k_l^q\tilde{\mathbf{q}}_{v_l}^\top\boldsymbol{\omega}_{v_l}$, $\tau_{ij}\int_{t-\tau_{ij}}^t\dot{\mathbf{Q}}_{v_j}^\top\dot{\mathbf{Q}}_{v_j}ds \leq \tau\int_{t-\tau}^t\dot{\mathbf{Q}}_{v_j}^\top\dot{\mathbf{Q}}_{v_j}ds$, and $\dot{\mathbf{Q}}_{v_i}^\top\dot{\mathbf{Q}}_{v_i} = (1/4)\boldsymbol{\omega}_{v_i}^\top\mathbf{T}(\mathbf{Q}_{v_i})^\top\mathbf{T}(\mathbf{Q}_{v_i})\boldsymbol{\omega}_{v_i} = (1/4)\boldsymbol{\omega}_{v_i}^\top\boldsymbol{\omega}_{v_i}$, we obtain

$$\dot{V} \leq -\sum_{i=1}^n k_i^z\boldsymbol{\omega}_{v_i}^\top\boldsymbol{\omega}_{v_i} - \sum_{i=1}^n k_i^d\lambda_i{\tilde{\mathbf{q}}_i^e}^\top\tilde{\mathbf{q}}_i^e \quad (18)$$

where k_i^z is given in the theorem. Therefore, $\dot{V}$ is negative semi-definite, and we conclude that $\boldsymbol{\omega}_{v_i}, \tilde{\mathbf{q}}_i^e \in \mathcal{L}_2\cap\mathcal{L}_\infty$, and $\boldsymbol{\omega}_i^e \in \mathcal{L}_\infty$. Note that $\tilde{\mathbf{Q}}_{v_l}$, $\bar{\mathbf{Q}}_{v_{ij}}$, $\mathbf{Q}_i^e$ and $\tilde{\mathbf{Q}}_i^e$ are naturally bounded by the definition of the unit-quaternion. Also, we conclude from (4) and (10) respectively that $\boldsymbol{\omega}_i, \dot{\boldsymbol{\omega}}_{v_i} \in \mathcal{L}_\infty$. Furthermore, we can see from (7) that $\tilde{\boldsymbol{\omega}}_i^e \in \mathcal{L}_\infty$, which leads us to conclude that $\dot{\tilde{\mathbf{q}}}_i^e \in \mathcal{L}_\infty$. As a result, we conclude that $\boldsymbol{\omega}_{v_i} \to \mathbf{0}_3$ and $\tilde{\mathbf{q}}_i^e \to \mathbf{0}_3$ for $i \in \mathcal{N}$, and consequently, $\boldsymbol{\beta}_i \to \mathbf{0}_3$ and $\tilde{\mathbf{Q}}_i^e \to \pm\mathbf{Q}_I$ for $i \in \mathcal{N}$.

Exploiting the above results, we can verify from (15) that $\dot{\boldsymbol{\omega}}_i^e \in \mathcal{L}_\infty$, and consequently we know from the first time-derivative of (7) that $\dot{\tilde{\boldsymbol{\omega}}}_i^e \in \mathcal{L}_\infty$. This implies that $\ddot{\tilde{\mathbf{Q}}}_i^e \in \mathcal{L}_\infty$, and we conclude that $\dot{\tilde{\mathbf{Q}}}_i^e \to \mathbf{0}_4$ by Barbălat Lemma. As a result, we conclude from the first relation in (7) that $\tilde{\boldsymbol{\omega}}_i^e \to \mathbf{0}_3$, which leads us to conclude from (7) that $\boldsymbol{\omega}_i^e \to \mathbf{0}_3$, for $i \in \mathcal{N}$. Moreover, we can see from the first time-derivative of (15), with (5), that $\ddot{\boldsymbol{\omega}}_i^e \in \mathcal{L}_\infty$ since $\dot{\boldsymbol{\omega}}_{v_i}, \dot{\boldsymbol{\omega}}_i^e \in \mathcal{L}_\infty$. Invoking Barbălat Lemma we conclude that $\dot{\boldsymbol{\omega}}_i^e \to \mathbf{0}_3$ and (15), with (5), reduces to $k_i^p\mathbf{q}_i^e \to \mathbf{0}_3$ for $i \in \mathcal{N}$, which indicates that $\mathbf{Q}_i^e \to \pm\mathbf{Q}_I$ for $i \in \mathcal{N}$. In addition, we can verify from (4) that $\boldsymbol{\omega}_i \to \mathbf{0}_3$. Therefore, we conclude that each rigid body synchronizes its attitude to the attitude of its corresponding virtual system.

Using similar relations as in Appendix C, we have: $\bar{\mathbf{q}}_{v_{ij}} = \mathbf{q}_{v_{ij}} - \mathbf{T}^\top(\mathbf{Q}_{v_i})(\mathbf{Q}_{v_j}(t-\tau_{ij}) - \mathbf{Q}_{v_j})$. Therefore, (10) can be rewritten as: $\dot{\boldsymbol{\omega}}_{v_i} = -\alpha\bar{\mathbf{u}}_i - k_i^\omega\boldsymbol{\omega}_{v_i} - \sum_{j=1}^n k_{ij}\mathbf{q}_{v_{ij}} - \sum_{j=1}^n k_{ij}\mathbf{T}^\top(\mathbf{Q}_{v_i})\int_{t-\tau_{ij}}^t\dot{\mathbf{Q}}_{v_j}ds$. Since $\boldsymbol{\omega}_{v_i} \to \mathbf{0}_3$, we can verify from (3) that $\dot{\mathbf{Q}}_{v_i} \to \mathbf{0}_4$ for $i \in \mathcal{N}$. This with the fact that τ_{ij} is bounded leads us to conclude that: $\int_{t-\tau_{ij}}^t\dot{\mathbf{Q}}_{v_j}ds \to \mathbf{0}_3$. In addition, we know that $\mathbf{q}_{v_{ij}}$ and $\bar{\mathbf{u}}_i$ (in the case of $\alpha = 1$) are uniformly continuous since we have shown that $\boldsymbol{\omega}_{v_i} \in \mathcal{L}_\infty$ for $i \in \mathcal{N}$. Therefore, invoking the extended Barbălat Lemma[2] (which can be proved following similar arguments as in the proof of Barbălat Lemma in [24] for example), we can conclude that $\dot{\boldsymbol{\omega}}_{v_i} \to \mathbf{0}_3$, and hence we know from (10) and the above results that

$$\alpha\bar{\mathbf{u}}_i + \sum_{j=1}^n k_{ij}\mathbf{q}_{v_{ij}} \to \mathbf{0}_3, \quad \text{for } i \in \mathcal{N}. \quad (19)$$

[2]*Extended Barbălat Lemma*: Let $x(t)$ denote a solution to the differential equation; $\dot{x} = a(t) + b(t)$, with $a(t)$ a uniformly continuous function. Assume that $\lim_{t\to+\infty} x(t) = c$ and $\lim_{t\to+\infty} b(t) = 0$, with c a constant value. Then, $\lim_{t\to+\infty}\dot{x}(t) = 0$.

For the $\mathbf{LSP}$, $\alpha = 0$ and (19) reduces to

$$\sum_{j=1}^{n} k_{ij}\mathbf{q}_{v_{ij}} \to \mathbf{0}_3, \quad \text{for } i \in \mathcal{N} \tag{20}$$

and we can conclude using the result of Lemma 1 in [6] that $\mathbf{q}_{v_{ij}} \to \mathbf{0}_3$ if the communication graph is a tree. As a result, we know that $\mathbf{Q}_{v_{ij}} \to \pm\mathbf{Q}_I$ for all $i, j \in \mathcal{N}$, and therefore, all virtual systems synchronize their attitudes to the same constant final attitude. Since we have already shown that the attitude of each rigid body converges asymptotically to the attitude of its corresponding virtual system, we conclude that the $\mathbf{LSP}$ is solved.

In the case where $\alpha = 1$, $\mathbf{LFP}$, we consider the set of equations (19), and take the sum of all equations over $i \in \mathcal{N}$ to obtain: $k_l^q \tilde{\mathbf{q}}_{v_l} + \sum_{i=1}^n \sum_{j=1}^n k_{ij}\mathbf{q}_{v_{ij}} \to \mathbf{0}_3$. Using the symmetry property of the undirected communication graph, and the relation $\mathbf{q}_{v_{ij}} = -\mathbf{q}_{v_{ji}}$, one can easily verify that $\sum_{i=1}^n \sum_{j=1}^n k_{ij}\mathbf{q}_{v_{ij}} = \mathbf{0}_3$. As a result, we conclude that $\tilde{\mathbf{q}}_{v_l} \to \mathbf{0}_3$, and (19) reduces to (20). Therefore, similarly to the $\mathbf{LSP}$ case, we exploit the result of Lemma 1 in [6] to conclude that $\mathbf{Q}_{v_{ij}} \to \pm\mathbf{Q}_I$ for all $i, j \in \mathcal{N}$ if the communication graph is a tree. Since we have already shown that $\tilde{\mathbf{Q}}_{v_l} \to \pm\mathbf{Q}_I$, we conclude that all virtual systems synchronize their attitudes to the desired attitude $\mathbf{Q}_d$. Consequently, since each rigid body synchronizes its attitude to the attitude of its corresponding virtual system, we conclude that the $\mathbf{LFP}$ is solved.

To prove the last part of the theorem, and following similar steps as in [5], we can verify that (20) is equivalent to: $\sum_{i=1}^n \sum_{j=1}^n k_{ij}\eta_{v_i}(1-\eta_{v_{ij}}) \to 0$. Note that this relation holds at the limit, and therefore, it is clear that if there exists a time after which $\eta_{v_i} > 0$ (or $\eta_{v_i} < 0$) for $i \in \mathcal{N}$, then the only solution to (20) is $\eta_{v_{ij}} \to 1$ and $\mathbf{q}_{v_{ij}} \to \mathbf{0}_3$. Therefore, following similar analysis as above, we can conclude that the $\mathbf{LSP}$ and $\mathbf{LFP}$ are solved for any connected undirected communication graph.

Appendix B
Proof of Theorem 2

First, since the directed communication graph $\mathcal{G}$ is strongly connected, there exists a vector $\gamma := \text{col}(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$, with $\gamma_i > 0$ such that $\gamma^\top \mathbf{L} = 0$, where $\mathbf{L} := [l_{ij}] \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of the communication graph $\mathcal{G}$ defined as: $l_{ii} = \sum_{j=1}^n k_{ij}$ and $l_{ij} = -k_{ij}$, [20].

Consider the following Lyapunov-Krasovskii functional: $W = V_1 + V_3$, with V_1 is given in the proof of Theorem 1 and

$$V_3 = \sum_{i=1}^{n} 2\gamma_i(1-\eta_{v_i}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i k_{ij}\int_{t-\tau_{ij}}^{t} \mathbf{q}_{v_j}^\top \mathbf{q}_{v_j} ds$$

with $\gamma_i > 0$ being defined as above. The time derivative of V_3 evaluated along (3) with (11) gives

$$\begin{aligned}\dot{V}_3 = &-\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i k_{ij}\mathbf{q}_{v_i}^\top\left(\mathbf{q}_{v_i} - \mathbf{q}_{v_j}(t-\tau_{ij})\right)\\ &+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i k_{ij}\left(\mathbf{q}_{v_j}^\top\mathbf{q}_{v_j} - \mathbf{q}_{v_j}^\top(t-\tau_{ij})\mathbf{q}_{v_j}(t-\tau_{ij})\right).\end{aligned}$$

Since the directed communication graph is assumed to be strongly connected, we can show that $\sum_{i=1}^n \sum_{j=1}^n \gamma_i(k_{ij}/2)(\mathbf{q}_{v_i}^\top \mathbf{q}_{v_i} - \mathbf{q}_{v_j}^\top \mathbf{q}_{v_j}) = (1/2)\gamma^\top \mathbf{L}\mathbf{x} = 0$, with $\mathbf{x} := (\mathbf{q}_{v_1}^\top \mathbf{q}_{v_1}, \ldots, \mathbf{q}_{v_n}^\top \mathbf{q}_{v_n})^\top$. As a result, we obtain

$$\dot{V}_3 = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i k_{ij}\left\|\mathbf{q}_{v_i} - \mathbf{q}_{v_j}(t-\tau_{ij})\right\|^2. \tag{21}$$

Therefore, the time derivative of W evaluated along the dynamics (15) and (3), in view of (17) and (21), is negative semi-definite, and hence we conclude that $\boldsymbol{\omega}_i^e \in \mathcal{L}_\infty$ and $\tilde{\mathbf{q}}_i^e, (\mathbf{q}_{v_i} - \mathbf{q}_{v_j}(t-\tau_{ij})) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Note that $\mathbf{Q}_{v_i}$ and $\tilde{\mathbf{Q}}_i^e$ are naturally bounded by definition of the unit-quaternion. In addition, we can verify from (7), and the fact that $\boldsymbol{\beta}_i \in \mathcal{L}_\infty$, that $\dot{\tilde{\mathbf{q}}}_i^e \in \mathcal{L}_\infty$ since $\tilde{\boldsymbol{\omega}}_i^e \in \mathcal{L}_\infty$. Also, it should be noted from (11) that $\boldsymbol{\omega}_{v_i} \in \mathcal{L}_\infty$, which leads us to conclude from (3) that $\dot{\mathbf{q}}_{v_i} \in \mathcal{L}_\infty$. As a result, we conclude that $\tilde{\mathbf{q}}_i^e \to \mathbf{0}_3$, for $i \in \mathcal{N}$, and $(\mathbf{q}_{v_i} - \mathbf{q}_{v_j}(t-\tau_{ij})) \to \mathbf{0}_3$, for $(i,j) \in \mathcal{E}$. Consequently, we have $\boldsymbol{\beta}_i \to \mathbf{0}_3$, $\tilde{\mathbf{Q}}_i^e \to \pm\mathbf{Q}_I$, $\boldsymbol{\omega}_{v_i} \to \mathbf{0}_3$, and $\dot{\mathbf{q}}_{v_i} \to \mathbf{0}_3$ for $i \in \mathcal{N}$. Moreover, by noting that $(\mathbf{q}_{v_i} - \mathbf{q}_{v_j}(t-\tau_{ij})) = (\mathbf{q}_{v_i} - \mathbf{q}_{v_j} + \int_{t-\tau_{ij}}^t \dot{\mathbf{q}}_{v_j} ds)$, we know that $(\mathbf{q}_{v_i} - \mathbf{q}_{v_j}) \to \mathbf{0}_3$ for $i, j \in \mathcal{N}$, since the communication graph is strongly connected. This leads us to conclude that all virtual systems synchronize their attitudes to the same constant final attitude.

Exploiting the above results, and since $\boldsymbol{\omega}_{v_i}$ and $\dot{\boldsymbol{\omega}}_{v_i}$ are bounded, we can show following the same steps as in the proof of Theorem 1 that $\tilde{\boldsymbol{\omega}}_i^e \to \mathbf{0}_3$, $\boldsymbol{\omega}_i^e \to \mathbf{0}_3$, and $\dot{\boldsymbol{\omega}}_i^e \to \mathbf{0}_3$ for $i \in \mathcal{N}$. This leads us to conclude that $\boldsymbol{\omega}_i \to \mathbf{0}_3$ and $\mathbf{Q}_i^e \to \pm\mathbf{Q}_I$, from (4) and (15) respectively. Therefore, we conclude that each rigid body synchronizes its attitude to the attitude of its corresponding virtual system. Since we have shown that all virtual systems synchronize their attitudes, we conclude that the $\mathbf{LSP}$ is solved.

Appendix C
Proof of Lemma 1

Using the definition of the quaternion multiplication and the relative attitude in (16) with the expression of $\mathbf{T}(\mathbf{Q}_{v_i})$, we have: $\mathbf{q}_{v_{ij}} = \eta_{v_j}\mathbf{q}_{v_i} - (\eta_{v_i}\mathbf{I}_3 - \mathbf{S}(\mathbf{q}_{v_i}))\mathbf{q}_{v_j} = -\mathbf{T}(\mathbf{Q}_{v_i})^\top \mathbf{Q}_{v_j}$, where we have used the relation $\mathbf{S}(\mathbf{q}_{v_i})^\top = -\mathbf{S}(\mathbf{q}_{v_i})$. Similarly, from (8) we have $\bar{\mathbf{q}}_{v_{ij}} = -\mathbf{T}(\mathbf{Q}_{v_i})^\top \mathbf{Q}_{v_j}(t-\tau_{ij})$. Therefore, using the fact that $\boldsymbol{\omega}_{v_i}^\top \mathbf{T}(\mathbf{Q}_{v_i})^\top = 2\dot{\mathbf{Q}}_{v_i}^\top$, one can show that $\boldsymbol{\omega}_{v_i}^\top(\bar{\mathbf{q}}_{v_{ij}} - \mathbf{q}_{v_{ij}}) = -\boldsymbol{\omega}_{v_i}^\top \mathbf{T}(\mathbf{Q}_{v_i})^\top(\mathbf{Q}_{v_j}(t-\tau_{ij}) - \mathbf{Q}_{v_j}) = 2\dot{\mathbf{Q}}_{v_i}^\top \int_{t-\tau_{ij}}^t \dot{\mathbf{Q}}_{v_j} ds$. Using Young's inequality, we can show that: $2\dot{\mathbf{Q}}_{v_i}^\top \int_{t-\tau_{ij}}^t \dot{\mathbf{Q}}_{v_j} ds \leq \epsilon_{ij}\dot{\mathbf{Q}}_{v_i}^\top \dot{\mathbf{Q}}_{v_i} + 1/\epsilon_{ij}(\int_{t-\tau_{ij}}^t \dot{\mathbf{Q}}_{v_j} ds)^\top(\int_{t-\tau_{ij}}^t \dot{\mathbf{Q}}_{v_j} ds)$, for $\epsilon_{ij} > 0$. Without loss of generality, we take $\epsilon_{ij} = \epsilon$, for $i, j \in \mathcal{N}$. Also, Jensen's inequality leads us to write: $(\int_{t-\tau_{ij}}^t \dot{\mathbf{Q}}_{v_j} ds)^\top(\int_{t-\tau_{ij}}^t \dot{\mathbf{Q}}_{v_j} ds) \leq \tau_{ij}\int_{t-\tau_{ij}}^t \dot{\mathbf{Q}}_{v_j}^\top \dot{\mathbf{Q}}_{v_j} ds$. Finally, the result of the lemma is obtained by combining the above relations.

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Unscented Kalman Filter: Aspects and Adaptive Setting of Scaling Parameter

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***Abstract*—This technical note deals with the unscented Kalman filter for state estimation of nonlinear stochastic dynamic systems with a special focus on the scaling parameter of the filter. Its standard choice is analyzed and its impact on the estimation quality is discussed. On the basis of the analysis, a novel method for adaptive setting of the parameter in the unscented Kalman filter is proposed. The results are illustrated in a numerical example.**

***Index Terms*—Bayesian methods, nonlinear filters, state estimation, stochastic systems.**

I. Introduction

The problem of nonlinear recursive state estimation of discrete-time stochastic dynamic systems from noisy measured data has been a subject of considerable research interest for the last several decades. In this technical note, the discrete-time nonlinear stochastic system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{w}_k,\ k = 0, 1, 2, \ldots \tag{1}$$

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k,\ k = 0, 1, 2, \ldots \tag{2}$$

is considered, where the vectors $\mathbf{x}_k \in \mathbb{R}^{nx}$ and $\mathbf{z}_k \in \mathbb{R}^{nz}$ represent the immeasurable state of the system and measurement at time instant k, respectively, $\mathbf{f}_k : \mathbb{R}^{nx} \to \mathbb{R}^{nx}$, $\mathbf{h}_k : \mathbb{R}^{nx} \to \mathbb{R}^{nz}$ are known vector functions, and $\mathbf{w}_k \in \mathbb{R}^{nx}$, $\mathbf{v}_k \in \mathbb{R}^{nz}$ are independent state and measurement white noises. The probability density functions (pdfs) of the noises are supposed to be Gaussian with zero means and known covariance matrices $\mathbf{Q}_k$ and $\mathbf{R}_k$, i.e., $p_{\mathbf{w}_k}(\mathbf{w}_k) = \mathcal{N}\{\mathbf{w}_k : \mathbf{0}_{n_x \times 1}, \mathbf{Q}_k\}$ and $p_{\mathbf{v}_k}(\mathbf{v}_k) = \mathcal{N}\{\mathbf{v}_k : \mathbf{0}_{n_z \times 1}, \mathbf{R}_k\}$, respectively. The initial state $\mathbf{x}_0$ is supposed to have Gaussian distribution $p_{\mathbf{x}_0}(\mathbf{x}_0) = \mathcal{N}\{\mathbf{x}_0 : \bar{\mathbf{x}}_0, \mathbf{P}_0\}$ and is independent of the noises.

The general solution to the estimation problem is given by the Bayesian recursive relations (BRRs) for computation of probability density functions (pdfs) of the state conditioned by the measurements [1]. These pdfs provide a full description of the estimated state. The BRRs are assumed in the following form:

$$p(\mathbf{x}_k|\mathbf{z}^k) = \frac{p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{z}^{k-1})} \tag{3}$$

$$p(\mathbf{x}_{k+1}|\mathbf{z}^k) = \int p(\mathbf{x}_{k+1}|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}^k)d\mathbf{x}_k \tag{4}$$

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