Observer-based Iterative Learning Control for a Class of Time-Varying Nonlinear Systems

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***Abstract*—In this brief, we propose an observer-based iterative learning control (ILC) scheme for the tracking problem of a class of time-varying nonlinear systems. First, a state observer is derived for the system under consideration, and sufficient conditions for the boundedness and the convergence to zero of the estimation error are given. Thereafter, an iterative learning rule—based on the proposed state observer—ensuring the boundedness of the tracking error is derived. Moreover, it is shown that if the initial state variables are known, it is possible to obtain a perfect convergence to zero, over a finite tracking horizon, when the number of iterations tends to infinity. By associating a state observer with the ILC scheme it is possible to avoid the use of state and output time-derivative measurements which are generally necessary in contraction mapping based ILC design for nonlinear systems without zero relative degree.**

I. Introduction and Problem Formulation

Iterative learning control (ILC) has gained a large amount of interest in the recent few years.[1] In fact, several contributions have been made, since the work of [2], toward improving ILC performance and relaxing ILC design constraints. It is well known that, the use of the output time derivative and the knowledge of the state variables are two important issues in ILC design for continuous-time nonlinear systems (see, for example, [1], [3]–[5], [9], [10]). In this brief, we propose an observer-based ILC scheme for a class of time-varying nonlinear systems with relative degree of one. First, an asymptotically stable observer is derived for the system under consideration. Thereafter, an ILC algorithm, using only the estimated states, is derived to ensure the learning convergence. The proposed observer-based ILC scheme allows one to avoid the use of state and output time-derivative measurements which are generally necessary in contraction mapping based ILC design for nonlinear systems without zero relative degree.

The system under consideration is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\left(x(t),t\right)u(t) + \phi\left(x(t),u(t),t\right) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x \in \mathrm{R}^n$, $u \in \mathrm{R}^m$, and $y \in \mathrm{R}^p$ represent, respectively, the state vector, the control input, and the system output. Matrices A, B and C are with appropriate dimensions, ϕ is a vector-valued function and $t \in [0,T]$ is the time.

Suppose that (1) operates repeatedly over a finite-time interval $[0,T]$. To distinguish between the signals involved in (1), at each operation or iteration, we introduce an additional subscript k. In order to avoid any confusion between the time variable t and the iteration variable k, we rewrite system (1) at the iteration k as follows:

$$\begin{aligned} \dot{x}_k(t) &= Ax_k(t) + B\left(x_k(t),t\right)u_k(t) + \phi\left(x_k(t),u_k(t),t\right) \\ y_k(t) &= Cx_k(t). \end{aligned} \tag{2}$$

Now, the control problem can be formulated as follows. Consider (2) and suppose that the desired trajectory is differentiable and given by $y_d(t)$ over the time interval $[0,T]$. Our objective is to design an iterative control law $u_k(t)$, such that the output $y_k(t)$ converges to the desired output $y_d(t)$, for all $t \in [0,T]$, when $k \to \infty$.

For the design of our controller, the following assumptions are needed.

- A1) The admissible range for the control input u_k is given by $\|u_k\| \leq u_{\max}$, where $u_{\max}$ is a known value obtained from the system's physical limitations.
- A2) There exists a bounded control law $u_d(t)$, over $[0,T]$, i.e., $\|u_d\| \leq u_{\max}$, such that $y_d(t) = Cx_d(t)$ and $\dot{x}_d(t) = Ax_d(t) + B(x_d(t),t)u_d(t) + \phi(x_d(t),u_d(t),t)$.
- A3) The matrix B is bounded and satisfies the Lipschitz condition with respect to x over the time interval $[0,T]$ (i.e., $\|B(.,.)\| \leq B_m$ and $\|B(x_1,t) - B(x_2,t)\| \leq K_B\|x_1 - x_2\|$, for any $(x_1,x_2) \in \mathbb{R}^{n\times n}$, where B_m and K_B are constant positive parameters). The function ϕ satisfies the Lipschitz condition with respect to x and u over the time interval $[0,T]$ (i.e., $\|\phi(x_1,u_1,t) - \phi(x_2,u_2,t)\| \leq K_{\phi,x}\|x_1 - x_2\| + K_{\phi,u}\|u_1 - u_2\|$ for any $(u_1,u_2) \in \mathbb{R}^{m\times m}$ and $(x_1,x_2) \in \mathbb{R}^{n\times n}$, where $K_{\phi,x}$ and $K_{\phi,u}$ are constant positive parameters.
- A4) $\mathrm{rank}(\mathrm{CB}(\cdot,\cdot)) = \mathrm{m}$.
- A5) The resetting condition is satisfied at each iteration, i.e., $x_k(0) = x_d(0)$, where $x_d(0)$ is the initial state corresponding to the desired trajectory.

Assumption (*A*1), which is not very restrictive from a practical point of view, is introduced for a technical reason guaranteeing the observer stability. Assumption (*A*2) defines all feasible trajectories that could be tracked with the admissible inputs such as defined in (*A*1). The Lipschitz conditions in assumption (*A*3) are classical in observers design and ILC design for nonlinear systems. They allow to avoid finite escape-time phenomena. The boundedness of $B(.,.)$ in (*A*3) is needed to establish the ILC convergence. This condition is realistic since ILC operates generally over a finite time interval. Assumption (*A*4) is a standard assumption in ILC design which guarantees the existence of the learning gain. Finally, Assumption (*A*5), which is also classical in ILC design, allows to achieve perfect tracking. This assumption can be traded against "nonperfect tracking", i.e., convergence of the tracking error to a certain domain around zero depending on the size of the initial error.

Throughout the brief, we will use the following norms: $\|M\| = \max_{1\leq i\leq m}\{\sum_{j=1}^{n}|m_{ij}|\}$ for a given matrix $M = [m_{ij}] \in \mathbb{R}^{m\times n}$, and $\|V\| = \max_{1\leq i\leq m}|v_i|$ for a given vector $V = [v_1,\ldots,v_m]^T$. We will also use the Sup norm denoted by $\|*(t)\|_\infty = \sup_{t\in[0,T]}\|*(t)\|$, and the λ-norm denoted by $\|*(t)\|_\lambda = \sup_{t\in[0,T]}\{e^{-\lambda t}\|*(t)\|\}$.

II. Observer-Based ILC Design

In the sequel, the time argument t will be omitted where there is no matter to any confusion. Consider the following state observer:

$$\begin{aligned} \dot{\hat{x}}_k &= A\hat{x}_k + \hat{B}_k u_k + \hat{\phi}_k + L\left(y_k - \hat{y}_k\right) \\ \hat{y}_k &= C\hat{x}_k \end{aligned} \tag{3}$$

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[1]For general overview of existing ILC, the reader is referred to the survey paper [7].

where L is the observer gain to be designed later, $\hat{B}_k \triangleq B(\hat{x}_k(t), t)$ and $\hat{\phi}_k \triangleq \phi(\hat{x}_k(t), u_k(t), t)$. The estimation error dynamics is then given by

$$\dot{e}_k = (A - LC)e_k + (B_k - \hat{B}_k)u_k + \phi_k - \hat{\phi}_k \tag{4}$$

where $e_k \triangleq x_k - \hat{x}_k$, $B_k \triangleq B(x_k(t), t)$ and $\phi_k \triangleq \phi(x_k(t), u_k(t), t)$. Using the following Lyapunov function candidate:

$$V = e_k^T P e_k \tag{5}$$

where P is a symmetric positive definite matrix, we obtain

$$\dot{V} = e_k^T \left(P(A - LC) + (A - LC)^T P\right) e_k + 2e_k^T P \left((B_k - \hat{B}_k)u_k + \phi_k - \hat{\phi}_k\right). \tag{6}$$

According to assumptions A1 and A3, the second term of the right-hand side of the previous equation might be bounded as follows:

$$\begin{aligned} &\left\| 2e_k^T P \left((B_k - \hat{B}_k)u_k + (\phi - \hat{\phi})\right)\right\| \\ &\le 2\|e_k\|\|P\| \left(\|B_k - \hat{B}_k\| u_{\max} + \|\phi_k - \hat{\phi}_k\|\right) \\ &\le 2\|e_k\|^2 \|P\| (K_B u_{\max} + K_{\phi,x}) \\ &\le 2\gamma \|e_k\|^2 \|P\| \\ &\le \gamma^2 e_k^T P P e_k + e_k^T e_k \end{aligned} \tag{7}$$

where, $\gamma = (K_B u_{\max} + K_{\phi,x})$.

Hence $\dot{V} \le e_k^T Q e_k$, with

$$Q = P(A - LC) + (A - LC)^T P + \gamma^2 PP + I. \tag{8}$$

According to [8], one has the following Lemma.

Lemma 1: If (A,C) is observable and if L is chosen such that $(A - LC)$ stable and

$$\min_{\omega \in \mathbb{R}^+} \sigma_{\min}(A - LC - j\omega I) > \gamma$$

where $\sigma_{\min}(*)$ denote the smallest singular value of $(*)$, then the observation error e_k is bounded and converges asymptotically to zero. □

Proof: See [8].

Now, let us consider the following iterative learning controller:

$$\begin{aligned} v_{k+1}(t) &= u_k(t) + G_k(t)\left(\dot{y}_d(t) - CA\hat{x}_k(t) - CB\left(\hat{x}_k(t), t\right)u_k(t) - C\phi\left(\hat{x}_k(t), u_k(t), t\right)\right) \\ u_{k+1}(t) &= \mathrm{sat}\left(\mathrm{v_{k+1}(t)}\right) \end{aligned} \tag{9}$$

where $\dot{y}_d(t)$ is the time derivative of $y_d(t)$, and $\hat{x}_k$ is given by the state observer (3). The initial control input is such that $\|u_0(t)\|_\infty \le u_{\max}$. For a vector $U = [u_1, \ldots, u_m]^T \in \mathbb{R}^m$, the saturation function is defined as

$$\mathrm{sat}(U) = [\mathrm{sat}(u_1), \ldots, \mathrm{sat}(u_m)]^T$$

with

$$\mathrm{sat}(u_i) = \begin{cases} u_i, & \text{if } |u_i| \le \overline{u}_i \\ \overline{u}_i, & \text{if } u_i > \overline{u}_i \\ -\overline{u}_i, & \text{if } u_i < -\overline{u}_i \end{cases}$$

for $i \in \{1, \ldots, m\}$, with $\|[\overline{u}_1, \ldots, \overline{u}_m]^T\| = u_{\max}$.

Let us denote the control error by $\tilde{u}_k = u_d - u_k$ and $\tilde{v}_k = u_d - v_k$. Hence

$$\begin{aligned} \tilde{v}_{k+1} &= u_d - v_{k+1} = \tilde{u}_k - (v_{k+1} - u_k) \\ &= \tilde{u}_k - G_k\left(\dot{y}_d - CA\hat{x}_k - CB(\hat{x}_k)u_k - C\phi(\hat{x}_k, u_k)\right) \\ &= \tilde{u}_k - G_k\left(CAx_d + CB(x_d)u_d + C\phi(x_d, u_d) - CA\hat{x}_k - CB(\hat{x}_k)u_k - C\phi(\hat{x}_k, u_k)\right) \\ &= \left(I - G_k CB(\hat{x}_k)\right)\tilde{u}_k - G_k CA\tilde{x}_k \\ &\quad - G_k C\left(B(x_d) - B(\hat{x}_k)\right)u_d \\ &\quad - G_k C\left(\phi(x_d, u_d) - \phi(\hat{x}_k, u_k)\right) \end{aligned} \tag{10}$$

where $\tilde{x}_k = x_d - \hat{x}_k$ and x_d is the state vector generated by the control input u_d. According to assumption A2 and the definition of u_{k+1} in (9), one can conclude that $\|\tilde{u}_{k+1}\| \le \|\tilde{v}_{k+1}\|$. Using the previous fact and taking the norm of (10), in view of A2 and A3, we obtain

$$\|\tilde{u}_{k+1}\| \le \left(\|I - G_k CB(\hat{x}_k)\| + K_{\phi,u}\|G_k C\|\right)\|\tilde{u}_k\| + \alpha\|G_k C\|\|\tilde{x}_k\| \tag{11}$$

where $\alpha = \|A\| + K_B\|u_d\|_\infty + K_{\phi,x}$. Now, using system (2) and the state observer (3), one has

$$\tilde{x}_k(t) = \tilde{x}_k(0) + \int_0^t \left(A\tilde{x}_k + B(x_d)u_d - B(\hat{x}_k)u_k + \phi(x_d, u_d) - \phi(\hat{x}_k, u_k) - LCe_k\right) d\tau$$

which, according to A3, leads to

$$\|\tilde{x}_k\| \le \|\tilde{x}_k(0)\| + \int_0^t \left(\alpha\|\tilde{x}_k\| + \beta_1\|\tilde{u}_k\| + \beta_2\|e_k\|\right) d\tau \tag{12}$$

where $\beta_1 = K_{\phi,u} + \|B(\hat{x}_k, t)\|_\infty = K_{\phi,u} + B_m$ and $\beta_2 = \|LC\|$. Applying the Bellman-Gronwall Lemma to (12) we obtain

$$\|\tilde{x}_k\| \le \|\tilde{x}_k(0)\| e^{\alpha t} + \int_0^t \left(\beta_1\|\tilde{u}_k\| + \beta_2\|e_k\|\right) e^{\alpha(t-\tau)} d\tau. \tag{13}$$

Now, from (13) and (11), one has

$$\|\tilde{u}_{k+1}\| \le \left(\|I - G_k CB(\hat{x}_k)\| + K_{\phi,u}\|G_k C\|\right)\|\tilde{u}_k\| + \alpha\|G_k C\| \int_0^t \left(\beta_1\|\tilde{u}_k\| + \beta_2\|e_k\|\right) e^{\alpha(t-\tau)} d\tau + \alpha\|G_k C\|\|\tilde{x}_k(0)\| e^{\alpha t}. \tag{14}$$

Multiplying the previous inequality by $e^{-\lambda t}$, $\lambda > \alpha$, and applying the λ-norm, we obtain

$$\begin{aligned} \|\tilde{u}_{k+1}\|_\lambda &\le \left(\|I - G_k CB(\hat{x}_k)\|_\infty + K_{\phi,u}\|G_k C\|_\infty\right)\|\tilde{u}_k\|_\lambda \\ &\quad + \sup_{t\in[0,T]} \left\{\alpha\|G_k C\|\,\|\tilde{x}_k(0)\|\, e^{(\alpha-\lambda)t}\right\} \\ &\quad + \sup_{t\in[0,T]} \left\{\alpha\|G_k C\| \int_0^t e^{-\lambda\tau}\left(\beta_1\|\tilde{u}_k\| + \beta_2\|e_k\|\right) \times e^{(\alpha-\lambda)(t-\tau)} d\tau\right\} \\ &\le \left(\|1 - G_k CB(\hat{x}_k)\|_\infty + K_{\phi,u}\|G_k C\|_\infty\right)\|\tilde{u}_k\|_\lambda \\ &\quad + \sup_{t\in[0,T]} \left\{\alpha\|G_k C\| \int_0^t \beta_1 e^{(\alpha-\lambda)(t-\tau)} d\tau\right\}\|\tilde{u}_k\|_\lambda \\ &\quad + \sup_{t\in[0,T]} \left\{\alpha\|G_k C\| \int_0^t \beta_2 e^{(\alpha-\lambda)(t-\tau)} d\tau\right\}\|e_k\|_\lambda \\ &\quad + \sup_{t\in[0,T]} \left\{\alpha\|G_k C\|\,\|\tilde{x}_k(0)\|\, e^{(\alpha-\lambda)t}\right\} \end{aligned} \tag{15}$$

which leads to

$$\|\tilde{u}_{k+1}\|_\lambda \le (\rho_1 + \rho_2)\|\tilde{u}_k\|_\lambda + \rho_3 \tag{16}$$

where

$$\rho_1 = \|I - G_k(t)CB\left(\hat{x}_k(t), t\right)\|_\infty + K_{\phi,u}\|G_k(t)C\|_\infty \tag{17}$$

$$\rho_2 = \frac{\alpha\beta_1}{\lambda - \alpha}\left(1 - e^{(\alpha-\lambda)T}\right)\|G_k(t)C\|_\infty \tag{18}$$

and

$$\rho_3 = \frac{\beta_2\rho_2}{\beta_1}\|e_k\|_\lambda + \alpha\|G_k(t)C\|_\infty\|\tilde{x}_k(0)\|. \tag{19}$$

If the control gain is such that $\|G_k(t)C\|_\infty$ is bounded, it is clear that there exists a sufficiently large λ making ρ_2 arbitrarily small. Thus, if $\rho_1 < 1$, there exists λ such that $\rho_1 + \rho_2 < 1$. Hence

$$\lim_{k\to\infty}\|\tilde{u}_k\|_\lambda \le \epsilon \tag{20}$$

with

$$\epsilon = \min\left\{2u_{\max}, \frac{\rho_3}{1 - \rho_1 - \rho_2}\right\}.$$

Now, let us evaluate the upper bound of the tracking error. In fact, we have

$$\delta x_k(t) = \delta x_k(0) + \int_0^t (A\delta x_k + B(x_d)u_d - B(x_k)u_k + \phi(x_d, u_d) - \phi(x_k, u_k))\, d\tau \qquad (21)$$

which leads to

$$\|\delta x_k(t)\| \leq \|\delta x_k(0)\| + \int_0^t (\alpha \|\delta x_k(\tau)\| + \beta_1 \|\tilde{u}_k(\tau)\|)\, d\tau \qquad (22)$$

where $\delta x_k = x_d - x_k$. Applying the Bellman-Gronwall Lemma, we obtain

$$\|\delta x_k(t)\| \leq \|\delta x_k(0)\| e^{\alpha t} + \int_0^t \beta_1 \|\tilde{u}_k(\tau)\| e^{\alpha(t-\tau)} d\tau. \qquad (23)$$

Multiplying the previous inequality by $e^{-\lambda t}$, $\lambda > \alpha$, and applying the λ-norm, we obtain

$$\begin{aligned} \|\delta x_k(t)\|_\lambda &\leq \sup_{t\in[0,T]} \left\{ \int_0^t \beta_1 e^{(\alpha-\lambda)(t-\tau)} d\tau \right\} \|\tilde{u}_k\|_\lambda + \sup_{t\in[0,T]} \left\{ \|\delta x_k(0)\| e^{(\alpha-\lambda)t} \right\} \\ &\leq \frac{\beta_1 \left(1 - e^{(\alpha-\lambda)T}\right)}{(\lambda - \alpha)} \|\tilde{u}_k\|_\lambda + \|\delta x_k(0)\|. \end{aligned} \qquad (24)$$

Now, using (20) and *A*5, one can conclude that

$$\lim_{k\to\infty} \|y_d(t) - y_k(t)\|_\lambda \leq \|C\| \frac{\beta_1 \left(1 - e^{(\alpha-\lambda)T}\right)}{(\lambda - \alpha)} \epsilon. \qquad (25)$$

If the condition in *Lemma 1* is fulfilled, then $\|e_k\|_\lambda$ is bounded. Hence, for a sufficiently large λ, $\rho_3 \to \alpha \|G_k(t)C\|_\infty \|\tilde{x}_k(0)\|$ when $k \to \infty$. Moreover, if $\tilde{x}_k(0) = 0$ then $\rho_3 \to 0$ and so does ϵ, which implies that the tracking error converges to zero when $k \to \infty$.

Now, one can summarize the previous development in the following theorem.

Theorem 1: Consider (2) with the iterative learning controller (9), where $\hat{x}$ is given by the state observer (3). Assume that the conditions in Lemma 1 are satisfied and let assumptions (A1-A5) be fulfilled. Then, the following hold.

i) If $\rho_1 < 1$, there exists $\lambda > \alpha$ such that

$$\lim_{k\to\infty} \|y_d(t) - y_k(t)\|_\lambda \leq \|C\| \frac{\beta_1 \left(1 - e^{(\alpha-\lambda)T}\right)}{(\lambda - \alpha)} \epsilon. \qquad (26)$$

ii) If $\rho_1 < 1$ and $\hat{x}_k(0) = x_k(0)$, then $\|y_d(t) - y_k(t)\|_\lambda$ is bounded and tends to zero when $k \to \infty$. □

III. Numerical Example

Consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 + 0.1e^{-t} sin(x_1) \\ \dot{x}_2 &= x_1 + 0.1 sin(x_2) + (0.05 + 0.05e^{-t})u \\ y &= x_2. \end{aligned} \qquad (27)$$

Thus

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad C = [0\ 1].$$

The admissible control input is such that $|u| \leq 1$. One can easily show that $K_{\phi,u} = 0$, $K_{\phi,x} = 0.1$, $B_m = 0.1$ and $K_B = 0.1$. Hence $\gamma = 0.2$. Taking $L = [\,L_1\ \ L_2\,]^T = [\,5\ \ 4\,]^T$, one can easily check that $A - LC$ is stable and $\min_{\omega\in\mathbb{R}^+} \sigma_{\min}(A - LC - j\omega I) = 0.7016 > \gamma$. Consequently, the Riccati equation

$$P(A - LC) + (A - LC)^T P + \gamma^2 PP + I = Q$$

with

$$Q = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}$$

gives the following symmetric positive definite matrix:

$$P = \begin{pmatrix} 23.9302 & -20.0153 \\ -20.0153 & 174.6246 \end{pmatrix}.$$

Considering the following reference trajectory $y_d(t) = 0.05t^2(1-t)$, $t \in [0, 1]$, and using the control law (9) with $G_k(t) = 1/(0.05 + 0.05e^{-t})$, and the state observer (3), with zero initial conditions, we obtain the result shown in Fig. 1.

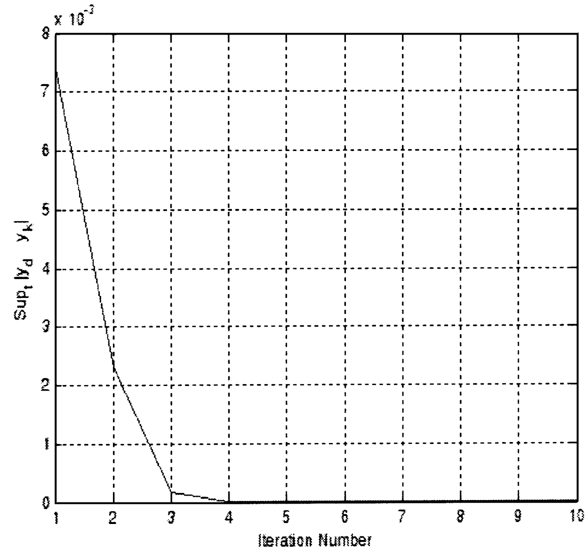

Fig. 1. Sup norm of the tracking error versus the number of iterations.

IV. Concluding Remarks

An observer-based iterative learning controller is proposed for a class of time-varying nonlinear systems. Sufficient conditions for the convergence of the observation error and the tracking error are derived. The following points should be pointed out.

1) The time derivative of the regulated output and the state variables are not needed as long as an adequate state observer can be designed for the system under consideration.
2) In this brief, we tried to consider a wide class of nonlinear systems by allowing u_k to appear in the function ϕ in a nonaffine manner. According to this fact, the sufficient condition $\rho_1 < 1$ appears to be somehow complicated than usual. However, if u_k appears just in an affine manner in the differential equation, i.e., the argument u_k does not appear in the function ϕ, then $K_{\phi,u} = 0$ and consequently the convergence condition reduces to $\rho_1 = \|I - G_k(t)CB(\hat{x}_k(t), t)\|_\infty < 1$. In this case the existence of G_k is guaranteed by assumption *A*4.
3) In some cases, the parameter γ can be large and this will make the condition in Lemma1 difficult to satisfy. This problem is inherent to observers design for nonlinear Lipschitz systems [6]. In [8], the author proposes an algorithm searching for the observer gain L and the maximum value of γ, namely γ_1, for a given pair (*A*,*C*). The gain L obtained by this algorithm is locally optimum and guarantees the observer stability for all value of γ less than γ_1.
4) The price to pay for the result developed in this brief is related to the observer design, which is not an easy task for nonlinear uncertain systems, and robust estimation techniques will be needed.

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Controllability of Linear Descriptor Systems

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***Abstract*—This brief studies the controllability of linear descriptor systems with multiple time delays in control. Several controllability concepts are investigated. First, necessary and sufficient criteria for the controllability of the canonical system are established. Then, equivalent criteria for that of the general system are given. Finally, it is pointed out that the controllability is independent of the size of the time delays.**

***Index Terms*—Controllability, descriptor systems, singular systems, time delay.**

I. INTRODUCTION

In recent years, there has been increasing interest in the analysis and synthesis of descriptor systems, or singular systems, due to their significance both in theory and applications [2]–[13].

For a linear time-invariant descriptor system, there are several controllability concepts with different meanings. For instance, the system is called *completely controllable (C controllable)* [4], if one can reach any terminal state from any admissible initial state; the system is called *R controllable* [4], if one can reach any terminal state in the reachable set from any admissible initial state; the system is called *impulse controllable (I controllable)* [12], if for every initial condition there exist a smooth(impulse-free) control $u(t)$ and a smooth state trajectory $x(t)$ solution; and the system is called *strongly controllable (S controllable)* [2], if it is both R controllable and I controllable.

Wei and Song discussed the controllability of descriptor systems with single time delay in control [13]. From the definition of the controllability they give, we can find that it corresponds to the C controllability of descriptor systems without time delay. Some necessary and sufficient conditions are also given in [13]. In this brief, we consider the more general case where the system contains multiple time delays in control. And we try to investigate all kinds of controllability of linear descriptor systems. Thus, the results given in [13] will become special cases of our results.

This brief is organized as follows. Section II formulates the problem and presents the preliminary results. Section III discusses the concept of reachability. Section IV discusses the controllability of the canonical system. Section V discusses the controllability of the general descriptor system. Finally, we provide the conclusion in Section VI.

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II. PROBLEM FORMULATION AND PRELIMINARIES

A. Problem Formulation

Consider a linear descriptor system with multiple time delays in control given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + \sum_{m=1}^{N} D_m u(t-h_m), \qquad t \geq 0 \\ x(0) &= x_0 \\ u(t) &= u_0(t), \qquad t \in [-h_N, 0) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^p$ the input vector; $E, A \in \Re^{n\times n}$, E is a singular matrix; $\det(sE-A) \not\equiv 0$; $B, D_1, \cdots, D_N \in \Re^{n\times p}$; $0 < h_1 < \cdots < h_N < \infty$ are N constant time delays; and $u_0(t)$ the initial control function.

For $\det(sE-A) \not\equiv 0$, there exist nonsingular matrices $P, Q \in \Re^{n\times n}$, such that (1) can be transformed into an equivalent canonical system

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + B_1u(t) + \sum_{m=1}^{N} D_{1,m}u(t-h_m), \qquad t \geq 0 \\ A2\dot{x}_2(t) &= x_2(t) + B_2u(t) + \sum_{m=1}^{N} D_{2,m}u(t-h_m), \qquad t \geq 0 \\ x_1(0) &= x_{10} \\ x_2(0) &= x_{20} \\ u(t) &= u_0(t), \qquad t \in [-h_N, 0) \end{aligned} \tag{2}$$

where $x_1 \in \Re^{n_1}$, $x_2 \in \Re^{n_2}$, $n_1 + n_2 = n$, $QEP = \text{diag}(I_1, N)$, $QAP = \text{diag}(A_1, I_2)$, $QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $QD_m = \begin{bmatrix} D_{1,m} \\ D_{2,m} \end{bmatrix}$, $m = 1, \cdots, N$; I_1, $A_1 \in \Re^{n_1\times n_1}$, I_1 is a unit matrix; A_2, $I_2 \in \Re^{n_2\times n_2}$, A_2 is a nilpotent matrix, I_2 is a unit matrix; B_1, $D_{1,m} \in \Re^{n_1\times p}$, B_2, $D_{2,m} \in \Re^{n_2\times p}$, $m = 1, \cdots, N$; $n_1 = \deg\det(sE-A)$.

In the remaining of the brief, we denote $D_1 = [D_{1,1}, \cdots, D_{1,N}]$, $D_2 = [D_{2,1}, \cdots, D_{2,N}]$ and $D = [D_1, \cdots, D_N]$. Let L be the index of E, then for (2), we have $A_2^{L-1} \neq 0$ and $A_2^L = 0$. Denote U the set of functions piecewise-differentiable $L-1$ times. As usual, we assume that the initial input $u_0 \in U$ and all the control input $u \in U$.

B. Preliminaries

In this subsection, we give some mathematical definitions and lemmas as the basic tools in the following discussion.