

Position Control of VTOL UAVs using Inertial Vector Measurements^{*}

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Abstract: Existing position controllers for VTOL UAVs require measurements of the system attitude (orientation) for feedback. However, in practice the orientation cannot be measured directly; it is rather obtained through the use of an attitude observer relying on a set of inertial vector measurements. In this paper a new control strategy is proposed for VTOL-UAVs which avoids the direct measurement of the system attitude (in terms of a rotation matrix, unit-quaternion or other attitude parameterization). In the proposed controller inertial vector measurements are used instead of the system attitude. This eliminates the need for an attitude observer thereby reducing the overall complexity of implementing the closed loop system and avoiding errors that are associated with the attitude observer.

Keywords: VTOL, UAV, position control, vector measurements.

1. INTRODUCTION

The autonomous operation of vertical take-off and landing (VTOL) aircraft has been a popular area of interest since they are inherently suited for many applications such as surveillance, search and rescue, structure inspection, defence, etc., especially where human presence is difficult and/or hazardous. As a result the research community has seen substantial improvements in the design of controllers for these types of vehicles (see, for instance, Hauser et al. (1992), Frazzoli et al. (2000), Hamel et al. (2002), Aguiar and Hespanha (2003), Pfimlin et al. (2007), Hua et al. (2009), Abdessameud and Tayebi (2010) and ?). Unsurprisingly, a common characteristic of the existing position controllers is they all require several system states to be accurately known including the system position and velocity (measured using a global positioning system (GPS)), the body angular-velocity (measured using a gyroscope), and the vehicle orientation or attitude. However, there does not exist a sensor which directly measures the attitude of a rigid body. To address this problem an attitude observer is usually sought to recover the attitude of the vehicle. Some manufacturers have developed a number of so-called *orientation sensors*. However, these devices are still dependant on some observer or other attitude estimation scheme. Therefore, the orientation data that is provided to the control scheme is an approximation due to the errors associated with the attitude observers.

In light of this shortcoming, it is no surprise that the problem of attitude observation is very popular in the research community, especially in the area of autonomous aircraft. The attitude estimation problem has been addressed using a number of different methods, including optimization,

Kalman filtering and other observers/filters developed using classic nonlinear control design methods. The goal of these attitude observers is to recover the orientation of a rigid body with respect to an inertial frame using a set of inertial vectors which are measured in the body-fixed frame. In many cases the body-referenced angular velocity (measured using a gyroscope) is also required. The type of attitude parameterizations used by the existing methods varies, although in most cases a direct cosine (rotation) matrix or unit-quaternion is used. In particular the unit-quaternion is preferred since it is numerically more efficient than rotation matrices, and is well defined over the entire space, unlike Euler angle parameterization which contains a well-known singularity. However, the quaternion representation is an overparameterization of the real-space which introduces some other difficulties. Since the map from quaternion space to the real space is non-injective (two-to-one), this introduces a topological obstruction to achieving global asymptotic stability (for more details see Bhat and Bernstein (2000) and Koditschek (1988)). As a result, the term *almost global* is often used to characterize equilibrium solutions when quaternion representation is used.

In some cases, vector measurements are used to directly calculate (reconstruct) the orientation of a rigid body (without using a filter or observer), such as the TRIAD algorithm proposed in Shuster and Oh (1979), or other methods such as Reynolds (1998) and Metni et al. (2005). In situations where the rigid-body is subjected to rotational motion, the attitude reconstructions can be combined with the system angular velocity vector in a filter, which is commonly referred to as a *complementary filter*. Examples of such filters are found in Mahony et al. (2005), Hamel and Mahony (2006), Tayebi et al. (2007) and Mahony et al. (2008). In some cases, the attitude

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reconstruction algorithms have been removed and filters have been designed directly from the vector measurements, which have been discussed in Hamel and Mahony (2006), Mahony et al. (2008) and Tayebi et al. (2010). Kalman filter-based observers have also been proven to be popular especially in attitude estimation of satellites. Examples of Kalman filter based observers can be found in the survey paper Crassidis et al. (2007).

The main concern we address in this paper is that the two problems associated with the control of VTOL UAVs, namely the attitude estimation and attitude control problems, have been addressed separately due to the complexity of combining both tasks into a single control problem. As a result, there is currently no guarantee for stability (to our knowledge) for closed-loop systems using attitude observers with a controller that assumes the orientation is directly measured. In this paper we acknowledge the fact that the system orientation is not known. Instead of using the system orientation as feedback, we use the vector measurements (which would normally be applied to the attitude estimator) as additional inputs to the position controller. Using the proposed controller, we show that for an appropriate choice of control gains the system states are uniformly bounded and converge to some predefined trajectory for almost all initial conditions.

An important intermediate step in the control design is to specify a desired system acceleration, which is derived to force the system position and velocity errors to zero. Due to the underactuated nature of this type of vehicle, the acceleration of the system is dependant on the system attitude and thrust. Consequently, it is necessary to obtain an appropriate desired attitude and thrust which satisfies the required system acceleration that forces the position and velocity errors to zero. To calculate the value of the desired system attitude, we use an attitude extraction algorithm that has been previously used by ?. The remaining steps of the control design is to obtain a value for the control torque input (that is applied to the rotational dynamics of the system) which drives the actual system attitude to the desired value. During this process, we specify a virtual control law for the desired system acceleration which is bounded *a priori*. This is an attractive characteristic especially in the case where one of the vector measurements is obtained using an accelerometer, which could otherwise be affected by large demanded accelerations due to the controller.

2. BACKGROUND

2.1 Attitude Representation

Let \mathcal{I} denote an inertial frame of reference rigidly attached to the earth (assumed flat), and \mathcal{B} denote a frame of reference rigidly attached to the aircraft center of gravity (COG) in North-East-Down coordinates. To describe the rotation from $\mathcal{I} \rightarrow \mathcal{B}$ we use the quaternion $Q = (\eta, q)$, $\eta \in \mathbb{R}$, $q \in \mathbb{R}^3$, where Q belongs to the set of unit quaternion $\mathbb{Q} \in \mathbb{Q} := \{Q \in \mathbb{S}^3, \|Q\| = 1\}$, where \mathbb{S} denotes a three-dimensional sphere. For more details on the unit-quaternion (in addition to other forms of attitude representation) the reader is referred to Shuster (1993), Murray et al. (1994), and Hughes (1986). The unit norm

constraint of the quaternion implies $\eta^2 + q^T q = 1$. The rotation $\mathcal{I} \rightarrow \mathcal{B}$ can also be described using a direct cosine (rotation) matrix $R(\eta, q) \in SO(3)$ where $SO(3)$ is the special-orthogonal group $SO(3) := \{R \in \mathbb{R}^{3 \times 3}, \det R = 1, RR^T = R^T R = I_{3 \times 3}\}$. The rotation matrix $R(\eta, q)$ corresponding to the unit quaternion $Q = (\eta, q)$ can be determined using a particular form of the *Rodrigues rotation formula* given by

$$R(\eta, q) = I_{3 \times 3} + 2S(q)^2 - 2\eta S(q), \quad (1)$$

where $S(\cdot)$ is the *skew-symmetric matrix*

$$S(u) = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \quad (2)$$

and $u = [u_1, u_2, u_3]^T$. The set \mathbb{Q} forms a group with the quaternion product operation, denoted by \odot , with the quaternion inverse defined by $Q^{-1} = (\eta, -q)$ and identity-quaternion $Q = (1, \mathbf{0})$. Given $Q, P \in \mathbb{Q}$ where $P = (p_0, p)$ the quaternion product is defined by $Q \odot P = (p_0\eta - q^T p, \quad \eta p + p_0 q + S(q)p)$.

2.2 Attitude Dynamics

Let ω denote the *body-referenced angular velocity* of the frame \mathcal{B} with respect to \mathcal{I} (expressed in \mathcal{B}). The body referenced angular velocity is used to obtain the time derivative of the quaternion Q using the well-known relationship

$$\dot{Q} = \frac{1}{2} Q \odot (0, \omega) = \frac{1}{2} \left[\eta I_{3 \times 3} + S(q) \right] \omega. \quad (3)$$

Similarly, an expression for the time-derivative of the rotation matrix $R(\eta, q)$ is given by $\dot{R}(\eta, q) = -S(\omega) R(\eta, q)$.

2.3 Bounded Functions

Let $h(\cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denote a bounded, twice-differentiable function and let $\phi_h(u) := \frac{\partial}{\partial u} h(u)$ and $f_\phi(u, v) := \frac{\partial}{\partial u} \phi_h(u) v \forall u, v \in \mathbb{R}^3$, which satisfy the following properties:

$$\left. \begin{array}{l} u^T h(u) > 0 \\ 0 \leq \|h(u)\| < 1 \\ 0 < \|\phi_h(u)\| \leq 1 \\ \|f_\phi(u, v)\| \leq c_f \|v\| \end{array} \right\} \forall u \in \mathbb{R}^3, \|u\| \in (0, \infty), \quad (4)$$

where c_f is a positive constant. One function which satisfies these conditions is given by $h(u) = (1 + u^T u)^{-1} u$ which further yields

$$\phi_h(u) = (1 + u^T u)^{-3/2} (I_{3 \times 3} - S(u)^2), \quad (5)$$

$$f_\phi(u, v) = (1 + u^T u)^{-5/2} (3(S(u)^2 - I_{3 \times 3})vu^T + (1 + u^T u)(2S(u)S(v) - S(v)S(u))), \quad (6)$$

from which one can find the bound $c_f = 4/\sqrt{3}$.

2.4 System Model

Let $p, v \in \mathbb{R}^3$ denote the position and velocity, respectively, of the vehicle COG expressed in the inertial frame \mathcal{I} .

Let the unit quaternion $Q = (\eta, q) \in \mathbb{Q}$ represent the rotation between the frame \mathcal{I} and \mathcal{B} . Let $R = R(\eta, q) \in SO(3)$ denote the rotation matrix corresponding to the quaternion Q . The vehicle rotational inertia tensor is denoted by $I_b \in \mathbb{R}^{3 \times 3}$ which is expressed in the body frame \mathcal{B} . Using the framework outlined in section 2.2 we consider the following well-known model for a VTOL UAV (Abdessameud and Tayebi (2010), Pfimlin et al. (2007))

$$\begin{aligned} \dot{p} &= v, \\ \dot{v} &= \mu, \quad \mu = g\hat{z} - u_t R^T \hat{z}, \\ \dot{Q} &= \frac{1}{2} \begin{bmatrix} -q^T \\ \eta I_{3 \times 3} + S(q) \end{bmatrix} \omega \\ I_b \dot{\omega} &= -\dot{S}(\omega) I_b \omega + \tau \end{aligned} \quad (7)$$

where $u_t = T/m$, T is the system thrust, m is the system mass, $\hat{z} = \text{col}[0, 0, 1]$, g is the gravitational acceleration and $\tau \in \mathbb{R}^3$ is the torque that is applied to the rotational dynamics of the system. The control input of the system is defined as $u = [u_t, \tau]^T$. The system output is defined as $y = [p, v, \omega, b_1, b_2, \dots, b_n]^T$ where $b_i = R r_i$, and $r_i \in \mathbb{R}^3$, $i = 1, 2, \dots, n$ are a set of known vectors expressed in the inertial frame \mathcal{I} . Note that the system attitude Q, R is no longer assumed to be a known output of the system.

3. PROBLEM FORMULATION

Let $p_d(t)$ denote a time-varying desired reference trajectory. We place the following two assumptions regarding the trajectory p_d and the inertial vectors r_i :

Assumption 1. The reference trajectory $p_d(t)$ and its first four derivatives are bounded. Furthermore, there exists a positive constant δ_{pz} such that $\|\hat{z}^T \ddot{p}_d\| < \delta_{pz} < g$.

Assumption 2. There are at least two non-collinear vectors r_i which are known and constant in the inertial frame \mathcal{I} .

The reference trajectory is used to define the following error signals

$$e_p = p - p_d, \quad e_v = v - \dot{p}_d. \quad (8)$$

Our main objective is to define a control input u such that the states (e_p, e_v, ω) are bounded and $e_p, e_v \rightarrow 0$ as $t \rightarrow \infty$. In light of (7), the error dynamics are governed by

$$\dot{e}_p = e_v, \quad \dot{e}_v = \mu - \ddot{p}_d. \quad (9)$$

Due to the underactuated nature of the system, we are forced to control the system position and velocity using the signal μ . Therefore, we define μ_d as a virtual control law that asymptotically stabilizes the translational dynamics and define the error $e_\mu = \mu - \mu_d$ where now we consider the additional challenge of forcing $e_\mu \rightarrow 0$ and therefore $\mu \rightarrow \mu_d$. Given the desired acceleration μ_d , we wish to specify a *desired attitude* R_d, Q_d and thrust u_t , that forces the position and velocity errors to zero. Section 3.1 describes an attitude extraction algorithm that has been previously proposed in ?, which we use to specify a desired attitude Q_d based on the desired acceleration μ_d . Subsequently, the remaining steps of the control design is to define the control torque input τ which forces $Q \rightarrow Q_d$ and therefore $\mu \rightarrow \mu_d$, thereby forcing the position and velocity errors to zero.

3.1 Attitude Extraction

Given μ_d we wish to find the value of attitude R_d and system thrust u_t that satisfies

$$g\hat{z} - u_t R_d^T \hat{z} = \mu_d. \quad (10)$$

A solution to this problem, which has been proposed in ?, is provided as follows: Given μ_d where $\mu_d \notin L$,

$$L := \{\mu_d \in \mathbb{R}^3; \mu_d = \text{col}[0, 0, \mu_{d3}]; \mu_{d3} \in [g, \infty)\}, \quad (11)$$

then a value of the thrust u_t and attitude $Q_d = (\eta_d, q_d)$ which satisfies (10) is given by

$$u_t = \|\mu_d - g\hat{z}\|, \quad (12)$$

$$\eta_d = \left(\frac{1}{2} \left(1 + \frac{g - \hat{z}^T \mu_d}{\|\mu_d - g\hat{z}\|} \right) \right)^{1/2}, \quad (13)$$

$$q_d = \frac{1}{2\|\mu_d - g\hat{z}\|} S(\mu_d) \hat{z}. \quad (14)$$

The extracted attitude Q_d has the time derivative

$$\dot{Q}_d = \frac{1}{2} \begin{bmatrix} -q_d^T \\ \eta_d I_{3 \times 3} + S(q_d) \end{bmatrix} \omega_d, \quad (15)$$

where the *desired angular velocity* ω_d is given by

$$\omega_d = M(\mu_d) \dot{\mu}_d, \quad (16)$$

where $M(\mu_d) = (-4S(\mu_d)\hat{z}\hat{z}^T + 4\eta_d^2 u_t S(\hat{z}) + 2S(\mu_d) - 2\hat{z}^T \mu_d S(\hat{z})) S(\mu_d - g\hat{z})^2 / (4\eta_d^2 u_t^4)$.

4. CONTROL DESIGN

The first step in the control design is to specify the virtual control law μ_d that asymptotically stabilizes the translational dynamics, which we choose to be

$$\mu_d = \ddot{p}_d - \Gamma_v (k_p h(e_p) + k_v h(e_v)), \quad (17)$$

where $\Gamma_v = \Gamma_v^T > 0$, $k_p, k_v > 0$ and $h(\cdot)$ is the bounded function defined in (2.3). Provided that the acceleration of the reference trajectory is bounded, (17) is bounded *a priori*. To avoid the singularity (11) we must also ensure that $\|\hat{z}^T \mu_d\| < g$. In light of Assumption 1, to avoid the singularity (11) we can place the following restriction

$$\delta_{pz} + (k_p + k_v) \|\hat{z}^T \Gamma_v\| < g, \quad (18)$$

which ensures a solution for the thrust u_t and desired attitude R_d exists, as given by (12)-(14). In light of (7), (8) and (17) the dynamics of the position and velocity error are given by

$$\dot{e}_p = e_v, \quad \dot{e}_v = -k_p \Gamma_v h(e_p) - k_v \Gamma_v h(e_v) + e_\mu. \quad (19)$$

Using the extraction method provided in section 3.1 and the value of μ_d from (17) we obtain the required system thrust u_t and the desired attitude Q_d . Since Assumption (1) is satisfied, there exists a positive constant δ_p such that $\|\ddot{p}_d\| < \delta_p$. Furthermore, (12) and (18) ensures that the thrust is positive and bounded such that

$$0 < \underline{c}_t < u_t < \bar{c}_t, \quad (20)$$

where $\bar{c}_t = g + \delta_p + (k_p + k_v) \|\Gamma_v\|$ and $\underline{c}_t = g - \delta_{pz} - (k_p + k_v) \|\hat{z}^T \Gamma_v\|$. Also, due to the lower bound of the

thrust specified by (20), in ? the authors show that the matrix $M(\mu_d)$ (which is used to calculate the desired angular velocity ω_d from (16)) has an upper-bound defined by

$$\|M(\mu_d)\| \leq \sqrt{2}/\underline{c}_t. \quad (21)$$

Using the desired quaternion Q_d obtained using (13)-(14) we obtain the corresponding desired rotation matrix from (1) which we denote as $R_d = R(\eta_d, q_d)$. Using the rotation matrix R_d and the n known inertial vectors r_i we define the *desired vector measurements* as

$$b_i^d = R_d r_i \quad i = 1, 2, \dots, n. \quad (22)$$

The *attitude error*, or the error between the actual and desired orientation, is defined using the unit-quaternion $Q_e = (\eta_e, q_e)$ and rotation matrix R_e by $Q_e = Q \odot Q_d^{-1}$ and $R_e = R_d^T R$, which have the time derivatives

$$\dot{Q}_e = \frac{1}{2} \begin{bmatrix} -q_e^T \\ \eta_e I_{3 \times 3} + S(q_e) \end{bmatrix} \omega_e, \quad \dot{R}_e = -S(\omega_e) R_e, \quad (23)$$

$$\omega_e = R_d^T (\omega - \omega_d),$$

where $\omega_d = M(\mu_d)\dot{\mu}_d$ is obtained using (16) and differentiating (17) (which is not exactly known since it depends on the system attitude (Q, R)). At this stage in the procedure, our objective is to force the actual system attitude to the desired attitude $R \rightarrow R_d$ using ω , which is equivalent to $R_e \rightarrow I_{3 \times 3}$ or $Q_e \rightarrow (\pm 1, 0)$. However, since ω is a state we define the virtual control law $\bar{\omega}$ that forces $R \rightarrow R_d$ and define the error $e_\omega = \omega - \bar{\omega}$ where we choose $\bar{\omega}$ to be

$$\bar{\omega} = M(\mu_d) f_{\mu_d} + \sum_{i=1}^n \gamma_i S(b_i^d) b_i, \quad (24)$$

where $f_{\mu_d} = p_d^{(3)} + k_p k_v \Gamma_v \phi_h(e_v) \Gamma_v h(e_p) - k_p \Gamma_v \phi_h(e_p) e_v$. The time derivative of $\bar{\omega}$ is given by $\dot{\bar{\omega}} = f_{\bar{\omega}} + g_{\bar{\omega}} e_\mu$ where

$$f_{\bar{\omega}} = Z(\mu_d, f_{\mu_d}) \left(f_{\mu_d} + k_v^2 \Gamma_v \phi_h(e_v) \Gamma_v h(e_p) \right) + M(\mu_d) \left(p_d^{(4)} + k_p k_v \Gamma_v \phi_h(e_v) \Gamma_v \phi_h(e_p) e_v - k_p k_v \Gamma_v f_\phi(e_v, \Gamma_v h(e_p)) \Gamma_v (k_p h(e_p) + k_v h(e_v)) - k_p \Gamma_v f_\phi(e_p, e_v) e_v + k_p^2 \Gamma_v \phi_h(e_p) \Gamma_v h(e_p) \right) + k_p k_v \Gamma_v \phi_h(e_p) \Gamma_v h(e_v) + \left(\sum_{i=1}^n \gamma_i S(b_i^d) S(b_i) \right) \omega - \left(\sum_{i=1}^n \gamma_i S(b_i) S(b_i^d) \right) \left(M(\mu_d) f_{\mu_d} + k_v^2 M(\mu_d) \Gamma_v \phi_h(e_v) \Gamma_v h(e_p) \right), \quad (25)$$

$$g_{\bar{\omega}} = -k_v Z(\mu_d, f_{\mu_d}) \Gamma_v \phi_h(e_v) - k_p M(\mu_d) \Gamma_v \phi_h(e_p) + k_p k_v M(\mu_d) \Gamma_v f_\phi(e_v, \Gamma_v h(e_p)) + k_v \left(\sum_{i=1}^n \gamma_i S(b_i) S(b_i^d) \right) M(\mu_d) \Gamma_v \phi_h(e_v), \quad (26)$$

$$Z(\mu_d, v) = \gamma_m \left(\mu_d^T v S(\hat{z}) + S(\hat{z}) \mu_d v^T - \hat{z}^T v S(\hat{z}) (2\mu_d \mu_d^T / u_t + (g + 2u_t) I_{3 \times 3} - 2g\mu_d \hat{z}^T / u_t) + S(\hat{z}) v \hat{z}^T (2u_t I_{3 \times 3} + (\mu_d - g\hat{z})(\mu_d - g\hat{z})^T / u_t + \mu_d (\mu_d - g\hat{z})^T / u_t) - 2S(\hat{z}) v (\mu_d - g\hat{z})^T + S(v) \mu_d (\mu_d - g\hat{z})^T / u_t + u_t S(v) \right) - \gamma_M^2 \left(S(\hat{z}) \mu_d \mu_d^T + (g + 2u_t) S(\mu_d) \hat{z} \hat{z}^T - u_t S(\mu_d) + (\hat{z}^T \mu_d u_t - 2\eta_d^2 u_t^2) S(\hat{z}) \right) v \left(-u_t^2 \hat{z}^T + 3u_t (\mu_d - g\hat{z})^T - 2\hat{z}^T (\mu_d - g\hat{z}) (\mu_d - g\hat{z})^T \right), \quad (27)$$

where $\gamma_M = u_t^{-2} (u_t + \hat{z}^T (g - \mu_d))^{-1}$. Finally, the proposed control torque input is defined as

$$\tau = S(\omega) I_b \omega + I_b f_{\bar{\omega}} - K_\omega e_\omega \quad (28)$$

where $K_\omega = K_\omega^T > 0$.

4.1 Main Result

Theorem 1. Consider the system defined by (7) where Assumptions 1 and 2 are satisfied, the system thrust u_t is defined by (12) using the virtual control law (17) under the restriction (18), and the torque control input τ is defined by (28), then for any initial condition $\eta_e(t_0) \neq 0^1$ there exists gains $\bar{\gamma}_i(\eta_e(t_0))$, $i = 1, 2, \dots, n$, such that for $\gamma_i > \bar{\gamma}_i$ the system states e_p, e_v, e_ω are bounded and $\lim_{t \rightarrow \infty} [e_p(t), e_v(t), e_\omega(t)] = \mathbf{0}$.

Proof. Consider the following Lyapunov function candidate

$$V = k_p \left(\sqrt{1 + \|e_p\|^2} - 1 \right) + \frac{1}{2} e_v^T \Gamma_v^{-1} e_v + \gamma_q (1 - \eta_e^2) + \frac{1}{2} e_\omega^T I_b e_\omega. \quad (29)$$

From (23) and using the properties $S(R_d r_i) = R_d S(r_i) R_d^T$, $S(q_e)^2 = q_e q_e^T - q_e^T q_e I_{3 \times 3}$ and $q_e^T S(r_i) q_e = 0$ one can obtain the following time derivative for the attitude error

$$\dot{\eta}_e = \eta_e q_e^T W q_e + \frac{k_v^2}{2} q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) \Gamma_v h(e_v) - \frac{1}{2} q_e^T R_d^T e_\omega - \frac{k_v}{2} q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) e_\mu, \quad (30)$$

where $W = -\sum_{i=1}^n \gamma_i S(r_i)^2$. Note that due to Assumption 2, the matrix W is positive definite and the eigenvalues of W can be arbitrarily increased using the gains γ_i . In light of (7), (19), (25), (26), (28) and (30) the time-derivative of (29) is given by

$$\dot{V} = -k_v e_v^T h(e_v) - 2\gamma_q \eta_e^2 q_e^T W q_e - e_\omega^T K_\omega e_\omega + e_v^T \Gamma_v^{-1} e_\mu - e_\omega^T I_b g_{\bar{\omega}} e_\mu + \gamma_q k_v \eta_e q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) e_\mu + \gamma_q \eta_e q_e^T R_d^T e_\omega - \gamma_q k_v^2 \eta_e q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) \Gamma_v h(e_v). \quad (31)$$

Due to (1), (7) and (10) the error signal $e_\mu = \mu - \mu_d$ can be expressed in terms of the vector part of the error quaternion, q_e , since $e_\mu = 2u_t (\eta_e I_{3 \times 3} - S(q_e)) S(R^T \hat{z}) q_e$. Therefore, due to (20) the signal e_μ is bounded by

$$\|e_\mu\| \leq 2c_t \|q_e\|. \quad (32)$$

We now focus our attention on the bound of $g_{\bar{\omega}}$. Due to the bound of the thrust (20), and the bound of μ_d due to (18) there exists a positive constant c_Z such that the matrix $Z(\mu_d, v)$ is bounded by

$$\|Z(\mu_d, v)\| \leq c_Z \|v\|. \quad (33)$$

Consequently, due to Assumption 1, (4), (21), (33) and the fact that $\|e_v\| \|\phi_h(e_v)\| \leq 1$, there exists a positive constant c_g such that the matrix $g_{\bar{\omega}}$ is bounded by

$$\|g_{\bar{\omega}}\| \leq c_g. \quad (34)$$

¹ This condition can be easily satisfied if the system is at rest at the time t_0 , i.e. $R(t_0) = I_{3 \times 3}$, since in this case $q = \mathbf{0}_{3 \times 1}$, $\eta = \pm 1$, and therefore $\eta_e = \pm \eta_d$. In this case, from (13) one can see that for any value of μ_d , $\inf |\eta_e(t_0)| = \inf |\eta_d(t_0)| = 1/\sqrt{2} > 0$.

Using this result in addition to (4), (21), (33), and (32) one can find $\|e_v\|^2 \leq 2\lambda_{\min}(\Gamma_v^{-1})^{-1}V(t)$, in addition to the following inequalities

$$e_v^T \Gamma_v^{-1} e_\mu \leq e_v^T e_v / (2\epsilon_1) + 2\epsilon_1 \bar{c}_t^2 \|\Gamma_v^{-1}\|^2 q_e^T q_e, \quad (35)$$

$$\gamma_q \eta_e q_e^T R_d^T e_\omega \leq \gamma_q \epsilon_2 \eta_e^2 q_e^T q_e / 2 + \gamma_q e_\omega^T e_\omega / (2\epsilon_2), \quad (36)$$

$$\begin{aligned} & \gamma_q k_v^2 \eta_e q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) \Gamma_v h(e_v) \\ & \leq \gamma_q k_v^2 c_\Gamma^4 \epsilon_3 \eta_e^2 q_e^T q_e / \underline{c}_t^2 + \gamma_q k_v^2 e_v^T h(e_v) / (2\epsilon_3), \end{aligned} \quad (37)$$

$$e_v^T I_b g_\omega e_\mu \leq 2c_g^2 \bar{c}_t^2 \|I_b\|^2 e_\omega^T e_\omega / \epsilon_4 + \epsilon_4 q_e^T q_e / 2, \quad (38)$$

$$\begin{aligned} & \gamma_q k_v \eta_e q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) e_\mu \\ & \leq 2\sqrt{2} \gamma_q k_v \bar{c}_t c_\Gamma |\eta_e| q_e^T q_e / \underline{c}_t, \end{aligned} \quad (39)$$

where $\epsilon_{1,2,3,4} > 0$, $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of (\cdot) and $c_\Gamma = \|\Gamma_v\|$. Consequently, the expression for \dot{V} can be rewritten as

$$\begin{aligned} \dot{V} & \leq -e_v^T e_\omega (\lambda_{\min}(K_\omega) - \gamma_q / (2\epsilon_2) - 2c_g^2 \bar{c}_t^2 \|I_b\|^2 / \epsilon_4) \\ & - e_v^T h(e_v) \left(k_v - \frac{1}{2\epsilon_1} \sqrt{2V(t) / \lambda_{\min}(\Gamma_v^{-1})} + 1 - \frac{\gamma_q k_v^2}{2\epsilon_3} \right) \\ & - 2\gamma_q \eta_e^2 q_e^T q_e \left(\lambda_{\min}(W) - \frac{1}{\eta_e^2} \frac{\epsilon_1 \bar{c}_t^2}{\gamma_q} \lambda_{\max}(\Gamma_v^{-1})^2 - \frac{\epsilon_2}{4} \right. \\ & \left. - \frac{k_v^2 c_\Gamma^4 \epsilon_3}{2\underline{c}_t^2} - \frac{1}{|\eta_e|} \frac{\sqrt{2} k_v \bar{c}_t c_\Gamma}{\underline{c}_t} - \frac{1}{\eta_e^2} \frac{\epsilon_4}{4\gamma_q} \right). \end{aligned} \quad (40)$$

In order to dominate some of the unwanted terms in (40) using the matrix W (and therefore ensure that $\dot{V} < 0$), we must show that a lower bound $\eta_e^* := \inf |\eta_e(t)| > 0$ exists. To further investigate this bound on η_e , we exclude the initial condition $\eta_e(t_0) = 0$, and consider the function $J = \eta_e^2 / 2$. In light of (30), the time derivative of J is given by

$$\begin{aligned} \dot{J} & = \frac{\eta_e^2}{2} \left(2q_e^T W q_e + \frac{k_v^2}{\eta_e} q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) \Gamma_v h(e_v) \right. \\ & \left. - \frac{1}{\eta_e} q_e^T R_d^T e_\omega - \frac{k_v}{\eta_e} q_e^T R_d^T M(\mu_d) \Gamma_v \phi_h(e_v) e_\mu \right). \end{aligned} \quad (41)$$

Using Young's inequality, in addition to the fact that $q_e^T q_e = 1 - \eta_e^2$ in addition to $e_v^T e_\omega \leq 2\bar{V}(t) / \lambda_{\min}(I_b)$ where $\bar{V}(t) = k_p(\sqrt{1 + \|e_p\|^2} - 1) + \frac{1}{2} e_v^T \Gamma_v^{-1} e_v + \gamma_q + \frac{1}{2} e_v^T I_b e_\omega$, we find \dot{J} is bounded by

$$\dot{J} \geq \eta_e^2 (1 - \eta_e^2) (\lambda_w - \rho / \eta_e^2 - \sigma(t) / (1 - \eta_e^2)), \quad (42)$$

$$\rho = \frac{1}{4} + \frac{k_v^2 c_\Gamma^4}{2\underline{c}_t^2} + \frac{\sqrt{2} k_v \bar{c}_t c_\Gamma}{\underline{c}_t}, \quad \sigma(t) = \frac{\bar{V}(t)}{2\lambda_{\min}(I_b)} + \frac{k_v^2}{4},$$

where $\lambda_w = \lambda_{\min}(W)$. Using (42) we wish to identify the region where $\dot{J} > 0$ and therefore $|\eta_e|$ is increasing. To find this region we consider a solution of $\inf \dot{J} = 0$ at the time t given by $\lambda_w = \rho / \eta_e^2 + \sigma(t) / (1 - \eta_e^2)$. Multiplying this result by η_e^2 and $1 - \eta_e^2$ we obtain

$$-\lambda_w \eta_e^4 + (\lambda_w + \rho - \sigma(t)) \eta_e^2 - \rho = 0. \quad (43)$$

Let $\alpha(t) = (\lambda_w + \rho - \sigma(t))^2 - 4\rho\lambda_w$. If $\alpha < 0$, (43) has complex roots and therefore the lower bound for \dot{J} is negative. Since we can find $\alpha(t) = \lambda_w(\lambda_w - 2(\rho + \sigma(t))) + (\sigma(t) - \rho)^2$, a simple, albeit conservative requirement to force $\alpha(t)$ to be positive is to take $\lambda_w > 2(\rho + \sigma(t))$. As

a result the solution to (43) has two real positive roots defined by

$$\bar{\eta}_e^2(t) = (\lambda_w + \rho - \sigma(t) + \sqrt{\alpha(t)}) / (2\lambda_w), \quad (44)$$

$$\underline{\eta}_e^2(t) = (\lambda_w + \rho - \sigma(t) - \sqrt{\alpha(t)}) / (2\lambda_w). \quad (45)$$

We define the open set $D := (\underline{\eta}_e, \bar{\eta}_e)$ (where we exclude the negative solutions for $\underline{\eta}_e$ and $\bar{\eta}_e$). Note that for any $|\eta_e(t)| \in D$ the value of \dot{J} is positive, and therefore $|\eta_e(t)|$ is increasing. Note the set D is time varying due to the value of $\sigma(t)$. However, if $\sigma(t)$ is a decreasing function and we choose

$$\lambda_w > 2(\rho + \sigma(t_0)), \quad (46)$$

we now will show that the lower limit $\underline{\eta}_e$ is decreasing, and the upper limit $\bar{\eta}_e$ is increasing with respect to t . To prove this relationship we consider the following partial derivative

$$\frac{\partial}{\partial \sigma(t)} \bar{\eta}_e^2 = \left(-1 - \sqrt{(\alpha(t) + 4\rho\lambda_w) / \alpha(t)} \right) / (2\lambda_w). \quad (47)$$

If $\sigma(t)$ is decreasing, then $\alpha(t)$ is increasing and therefore (47) is well-defined and negative. The partial derivative of the lower limit is given by

$$\frac{\partial}{\partial \sigma(t)} \underline{\eta}_e^2 = \left(-1 + \sqrt{(\alpha(t) + 4\rho\lambda_w) / \alpha(t)} \right) / (2\lambda_w), \quad (48)$$

which is always positive. Therefore, if $\sigma(t)$ is decreasing the value of $\underline{\eta}_e^2$ is decreasing, the value of $\bar{\eta}_e^2$ is increasing and the set D approaches $D \rightarrow (0, 1)$. The gain $\lambda_w = \lambda_{\min}(W) = \lambda_{\min}(-\sum_{i=1}^n \gamma_i S(r_i)^2)$ can be arbitrarily enlarged using the gains γ_i to ensure (46) is satisfied and therefore the domain D exists. Therefore, since $\lim_{\lambda_w \rightarrow \infty} \underline{\eta}_e^2 = 0$, there exists a value \bar{W}_1 such that for all $\lambda_{\min}(W) > \lambda_{\min}(\bar{W}_1)$, $0 < \underline{\eta}_e(t_0) < |\eta_e(t_0)|$, where we exclude the negative solution for $\underline{\eta}_e(t_0)$. Consequently, if $\sigma(t)$ is a decreasing function the minimum possible value for $\eta_e^* = \inf |\eta_e(t)|$ is given by $\eta_e^* = \min\{|\eta_e(t_0)|, \bar{\eta}_e(t_0)\}$, where we exclude the negative solution for $\bar{\eta}_e(t_0)$. The final step of the proof is to show that $V(t)$ and therefore $\sigma(t)$ are decreasing functions. If we recall the value of \dot{V} from (40), one can see that there exist values $\bar{\epsilon}_1$ and $\bar{\epsilon}_3$ such that for $\epsilon_1 > \bar{\epsilon}_1$ and $\epsilon_3 > \bar{\epsilon}_3$, the following inequality is satisfied

$$k_v > \sqrt{2V(t_0) / \lambda_{\min}(\Gamma_v^{-1})} + 1 / (2\epsilon_1) + \gamma_q k_v^2 / (2\epsilon_3), \quad (49)$$

for any $k_v > 0$. Furthermore, there exist values $\bar{\epsilon}_2$ and $\bar{\epsilon}_4$ such that for $\epsilon_2 > \bar{\epsilon}_2$, $\epsilon_4 > \bar{\epsilon}_4$

$$\lambda_{\min}(K_\omega) > \gamma_q / (2\epsilon_2) + 2c_g^2 \bar{c}_t^2 \|I_b\|^2 / \epsilon_4, \quad (50)$$

for any $K_\omega = K_\omega^T > 0$. Also, there exists a gain \bar{W}_2 such that for $\lambda_{\min}(W) > \lambda_{\min}(\bar{W}_2)$

$$\begin{aligned} \lambda_{\min}(W) & > \frac{1}{(\eta_e^*)^2} \frac{\epsilon_1 \bar{c}_t^2}{\gamma_q} \lambda_{\max}(\Gamma_v^{-1})^2 + \frac{\epsilon_2}{4} + \frac{k_v^2 c_\Gamma^4 \epsilon_3}{2\underline{c}_t^2} \\ & + \frac{1}{\eta_e^*} \frac{\sqrt{2} k_v \bar{c}_t c_\Gamma}{\underline{c}_t} + \frac{1}{(\eta_e^*)^2} \frac{\epsilon_4}{4\gamma_q}. \end{aligned} \quad (51)$$

There are two conditions for the gain W , where the minimum bound W_1 ensures that $\underline{\eta}_e \leq |\eta_e(t_0)|$, and the minimum bound W_2 which ensures that (51) is satisfied. There exists gains $\bar{\gamma}_i, i = 1, 2, \dots, n$, such that for all $\gamma_i > \bar{\gamma}_i$ $\lambda_{\min}(W) > \max\{\lambda_{\min}(\bar{W}_1), \lambda_{\min}(\bar{W}_2)\}$, which satisfies both requirements. Therefore, under this condition from (40) one can see that $\dot{V}(t_0) \leq 0$, which implies that for sufficiently small δ , $V(t_0 + \delta) \leq V(t_0)$, $\sigma(t_0 + \delta) \leq \sigma(t_0)$ and $|\eta_e(t_0 + \delta)| \geq \eta_e^*$. Since $V(t_0 + \delta) \leq V(t_0)$, $\sigma(t_0 + \delta) \leq \sigma(t_0)$ and $\eta_e(t_0 + \delta) \geq \eta_e^*$, the inequalities (49)-(51) remain satisfied, which implies $\dot{V}(t_0 + \delta) \leq 0$. Therefore, by induction the value of \dot{V} is guaranteed to be non-positive for all $t > t_0$ and

$$\dot{V} \leq -\delta_v e_v^T h(e_v) - 2\delta_q \eta_e^2 q_e^T q_e - \delta_\omega e_\omega^T e_\omega, \quad (52)$$

$$\delta_v = k_v - \frac{1}{2\epsilon_1} \sqrt{2\lambda_{\min}(\Gamma_v^{-1})^{-1} V(t_0) + 1} - \frac{\gamma_q k_v^2}{2\epsilon_3}, \quad (53)$$

$$\delta_\omega = \lambda_{\min}(K_\omega) - \gamma_q / (2\epsilon_2) - 2c_g^2 \bar{c}_t^2 \|I_b\|^2 / \epsilon_4, \quad (54)$$

$$\delta_q = \lambda_{\min}(W) - \frac{1}{(\eta_e^*)^2} \frac{\epsilon_1 \bar{c}_t^2}{\gamma_q} \lambda_{\max}(\Gamma_v^{-1})^2 - \frac{\epsilon_2}{4} - \frac{k_v^2 c_\Gamma^4 \epsilon_3}{2\bar{c}_t^2} - \frac{1}{|\eta_e^*|} \frac{\sqrt{2} k_v \bar{c}_t c_\Gamma}{\underline{c}_t} - \frac{1}{(\eta_e^*)^2} \frac{\epsilon_4}{4\gamma_q}. \quad (55)$$

Since \ddot{V} is bounded due to Assumption 1, Barbalat's Lemma implies that $[e_v, q_e, e_\omega] \rightarrow 0$, and since $\dot{e}_v \rightarrow 0$, $e_p \rightarrow 0$.

5. CONCLUSION

Existing controllers for the autonomous operation of VTOL UAVs require that the vehicles attitude or orientation be directly measured to be applied to the feedback loop. This requirement is usually not satisfied since an attitude observer or other estimation scheme is often implemented to recover the systems attitude based on a set of vector measurements. Since the attitude estimation and control problems are viewed separately, there are no guarantee for stability when the two systems are used together. To address this problem, a new position controller has been developed which uses the vector measurements directly, rather than assuming the vehicles orientation is measured, with accompanying proofs for stability. As an added benefit, implementation of the proposed controller would not require the use of an attitude observer, thus reducing the overall complexity of the closed-loop system.

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