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Some Optimization Aspects on the Lie Group SO(3) ⋆

Soulaimane Berkane * **Abdelhamid Tayebi** *,**

* *Department of Electrical and Computer Engineering, University of Western Ontario, London, Ontario, Canada, N6A 3K7 (e-mail: sberkane@uwo.ca).*
** *Department of Electrical Engineering, Lakehead University, Thunder Bay, Ontario, Canada, P7B 5E1 (e-mail: atayebi@lakeheadu.ca).*

Abstract: In this paper, we investigate the LQR-like optimal control problem on the special orthogonal group SO(3). Using the dynamic programming approach, we derive a Hamilton-Jacobi-Bellman equation in the general case where a generic distance on SO(3) is used in the cost functional. We show that the geodesic distance on SO(3) yields results analogous to the well known results for linear systems. Static and dynamic Riccati-like equations for both infinite and finite time-horizon optimal control problems are obtained.

Keywords: Optimal control; Special orthogonal group $SO(3)$; Geodesic distance; Riccati equations; Hamilton-Jacobi-Bellman equation.

1. INTRODUCTION

For linear systems, the problem of finding an optimal control input u minimizing a quadratic cost functional

$$J(t_0, x_0, u) = \int_{t_0}^{\infty} \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt, \quad (1)$$

subject to the dynamic constraint

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n,$$

is referred to as the *linear-quadratic regulator* (LQR). This is a fundamental problem in control theory and its solution is a linear state feedback $u = -Kx$ where $K = R^{-1}B^T P$, with P being a symmetric positive definite matrix solution of the algebraic Riccati equation [Kwakernaak and Sivan, 1972].

In this work we introduce an analogous LQR-like problem on the Lie group SO(3). To our best knowledge, the only paper that appears to tackle this problem is Saccon et al. [2010], where a solution to the optimal kinematic control problem on SO(3) is derived in the infinite time-horizon case with scalar penalties in the cost. Unfortunately, the use of the Euclidean distance on SO(3) as a measure of the energy of the state has considerably limited the results obtained in Saccon et al. [2010].
In the present work, we show that the geodesic distance on SO(3) associated to the natural Riemanian metric structure on SO(3) yields some interesting results that are analogous to those of linear systems. Static and dynamic Ricatti-like equations for the infinite and finite time-horizon cost functional with scalar and matrix penalties are obtained.

This paper is organized as follows. We start in section II with some mathematical tools and preliminaries that will be used throughout this paper. In section III, the kinematic optimal control on the Lie group SO(3) is formulated and a sufficient optimality condition is derived using the dynamic programming approach. This sufficient condition is given in terms of a Hamilton-Jacobi-Bellman (HJB) equation on SO(3) using a generic state energy function. The solution to this problem given in Saccon et al. [2010] is briefly reviewed and the shortcomings of using the Euclidean distance are discussed. In section IV, we show that the LQR-like problem on SO(3) can be naturally solved when considering the geodesic distance on SO(3). We show that the unique solution to the HJB equation is a quadratic function that has the same structure as the distance used in the cost functional. As in the LQR solution for linear systems, the optimal feedback is explicitly given as a function of the solution of an algebraic or dynamic Riccati equation depending on the type of the problem under consideration (infinite or finite time-horizon).

2. MATHEMATICAL PRELIMINARIES

2.1 Notations

The sets of real and nonnegative real are denoted as $\mathbb{R}$ and $\mathbb{R}^+$, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Given two matrices $A, B \in \mathbb{R}^{m\times n}$, their Euclidean inner product is defined as $\langle\langle A, B\rangle\rangle = \mathrm{tr}(A^T B)$, where $(\cdot)^T$ denotes the transpose of $(\cdot)$. The Frobenius norm of a matrix $A \in \mathbb{R}^{n\times m}$ is $||A|| = \sqrt{\langle\langle A, A\rangle\rangle}$. Given a manifold M, a tangent vector at x is $\gamma'(0)$ for some smooth path $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x$. The *tangent space* to M at x is the set of all tangent vectors at x, denoted $T_x M$. The disjoint union of all tangent spaces forms the *tangent bundle* TM. Let M and N be two smooth manifolds and let $f : M \to N$ be a differentiable map. The *tangent map*

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(differential) of f at a point $x \in M$ is the map [Darryl D. Holm and Stoica, 2009]

$$Df(x) : T_xM \to T_{f(x)}N$$
$$\xi \mapsto Df(x)\cdot\xi := \left.\frac{d\ (f\circ\gamma(\tau))}{d\tau}\right|_{\tau=0},$$

where $\gamma(\tau)$ is a path in M such that $\gamma(0) = x$ and $d\,\gamma(\tau)/d\tau|_{\tau=0} = \xi$. Let $f : M \to \mathbb{R}$ be a differentiable real-valued function. Let $\langle\ ,\ \rangle_x : T_xM \times T_xM \to \mathbb{R}$ be a *Riemannian metric* on M. The *gradient* of f, denoted $\nabla f(x) \in T_xM$, relative to the Riemannian metric $\langle\ ,\ \rangle_x$ is uniquely defined by

$$Df(x)\cdot\xi = \langle\nabla f(x),\xi\rangle_x \quad \text{for all } \xi \in T_xM.$$

2.2 The special orthogonal group of rotations $SO(3)$

Consider the *general linear group* $GL(3)$. A square matrix $R \in GL(3)$ is called a *rotation matrix* if R belongs to the *special orthogonal group* $SO(3) \subset GL(3)$ where

$$SO(3) := \{R \in \mathbb{R}^{3\times3} |\ \det(R) = 1,\ RR^T = I\},$$

and $I = I_{3\times3}$ is the three-dimensional identity matrix. The *Lie algebra* of $SO(3)$, denoted by $\mathfrak{so}(3)$, is the vector space of 3-by-3 skew-symmetric matrices

$$\mathfrak{so}(3) = \left\{\Omega \in \mathbb{R}^{3\times3}\ |\ \Omega^T = -\Omega\right\}.$$

The group $SO(3)$ has a compact manifold structure where its *tangent spaces* are identified by

$$T_RSO(3) := \{R\Omega \mid \Omega \in \mathfrak{so}(3)\}.$$

The Euclidean inner product on $\mathbb{R}^{3\times3}$, when restricted to the Lie-algebra of skew symmetric matrices, defines the following *left-invariant* Riemannian metric on $SO(3)$

$$\langle R\Omega_1, R\Omega_2\rangle_R := \langle\langle\Omega_1,\Omega_2\rangle\rangle, \tag{2}$$

for all $R \in SO(3)$ and $\Omega_1, \Omega_2 \in \mathfrak{so}(3)$. Let $\times$ denote the vector cross-product on $\mathbb{R}^3$ and define the map $[\ .\ ]_\times : \mathbb{R}^3 \to \mathfrak{so}(3)$; $\omega \mapsto [\omega]_\times$ such that

$$[\omega]_\times u = \omega \times u, \quad \text{for all } \omega, u \in \mathbb{R}^3.$$

Let vex : $\mathfrak{so}(3) \to \mathbb{R}^3$ denotes the inverse isomorphism of the map $[\ .\ ]_\times$, such that $\text{vex}([\omega]_\times) = \omega$, for all $\omega \in \mathbb{R}^3$ and $[\text{vex}(\Omega)]_\times = \Omega$, for all $\Omega \in \mathfrak{so}(3)$. The *adjoint operator* on $\mathfrak{so}(3)$ corresponding to $\Omega_1 \in \mathfrak{so}(3)$ is defined by

$$\text{ad}_{\Omega_1}\Omega_2 = [\Omega_1,\Omega_2], \qquad \Omega_2 \in \mathfrak{so}(3),$$

where $[\Omega_1,\Omega_2]$ denotes the Lie bracket operator (matrix commutator) given by

$$[\Omega_1,\Omega_2] = \Omega_1\Omega_2 - \Omega_2\Omega_1.$$

Let $\mathbb{P}_a(A)$ denote the projection of a square matrix $A \in \mathbb{R}^{3\times3}$ on the Lie algebra $\mathfrak{so}(3)$ of skew symmetric matrices given by

$$\mathbb{P}_a(A) := \frac{1}{2}(A - A^T).$$

For a given symmetric positive definite matrix $K = K^T > 0$, we define the positive definite symmetric operator $\Sigma_K : \mathfrak{so}(3) \to \mathfrak{so}(3)$ [Bloch et al., 2008]

$$\Sigma_K(\Omega) = \Omega K + K\Omega, \quad \Omega \in \mathfrak{so}(3).$$

Lemma 1. The operator Σ_K is an isomorphism and admits an inverse map $\Sigma_K^{-1} : \mathfrak{so}(3) \to \mathfrak{so}(3)$ given by

$$\Sigma_K^{-1}(.) = \Sigma_{\sigma(\rho(K)^{-1})}(.),$$

where

$$\sigma(A) := \frac{1}{2}\text{tr}(A)I - A, \quad \rho(A) = \text{tr}(A)I - A,$$

and $\rho(A)^{-1}$ is the inverse matrix of $\rho(A)$.

We associate to the operator $\Sigma_K(.)$ the *quadratic* function $\Psi_K : \mathfrak{so}(3) \to \mathbb{R}^+$ such that

$$\Psi_K(\Omega) := \frac{1}{2}\langle\langle\Omega, \Sigma_K(\Omega)\rangle\rangle. \tag{3}$$

The vector space $\mathfrak{so}(3)$ allows to represent elements of $SO(3)$ via an *exponential map* [Bullo and Lewis, 2005]

$$e^{[a]_\times} = \begin{cases} I, & a = 0 \\ I + \dfrac{\sin(||a||)}{||a||}[a]_\times + \dfrac{1-\cos(||a||)}{||a||^2}[a]_\times^2 & a \neq 0 \end{cases} \tag{4}$$

Equation (4) is referred to as Rodrigues Formula. The exponential map is a diffeomorphism between $\Pi_{\mathfrak{so}(3)} = \{[a]_\times \in \mathfrak{so}(3) \mid a \in \mathbb{R}^3, ||a|| < \pi\}$ and $\Pi_{SO(3)} = \{R \in SO(3) \mid \text{tr}(R) \neq -1\}$. The inverse map $\log : \Pi_{SO(3)} \to \Pi_{\mathfrak{so}(3)}$ is given by

$$\log(R) = \begin{cases} 0_{3\times3} & R = I \\ \dfrac{\theta(R)}{2\sin(\theta(R))}(R - R^T) & R \neq I \end{cases}$$

where $\theta : \Pi_{SO(3)} \to [0,\pi)$ is defined by

$$\theta(R) = \arccos\left(\frac{\text{tr}(R) - 1}{2}\right). \tag{5}$$

3. KINEMATIC OPTIMAL CONTROL ON SO(3)

A generic kinematic optimal control problem on the Lie group SO(3) can be formulated as follows. Given initial data (t_0, R_0), we consider the optimization problem

$$\min_{\Omega\in\mathfrak{so}(3)} J(t_0,R_0,\Omega) := \min_{\Omega\in\mathfrak{so}(3)} \int_{t_0}^{t_f} e^{-\gamma t}\mathcal{C}(R(t),\Omega(t))dt + e^{-\gamma t_f}\psi(R(t_f)), \tag{6}$$

subject to $\dot{R}(t) = R(t)\Omega(t)$, where $\mathcal{C}(R(t),\Omega(t))$ is an incremental cost, $\psi(R(t_f))$ is a terminal cost and $\gamma > 0$ corresponds to a discount factor. In particular, for a *quadratic* optimal control problem on $SO(3)$ the incremental cost takes the form

$$\mathcal{C}(R,\Omega) = \frac{1}{2}d^2_{SO(3)}(I,R) + \Psi_L(\Omega),$$

where $d_{SO(3)}(I,R)$ is some given distance on $SO(3)$, and L is a symmetric positive definite weighting matrix.

3.1 Hamilton-Jacobi-Bellman equation on SO(3)

In this section we derive sufficient optimality conditions using the dynamic programming approach. The following value function

$$V(t_0,R_0) := \inf_{\Omega\in\mathfrak{so}(3)} J(t_0,R_0,\Omega),$$

is the unique viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation [Bressan, 2003]

$$\frac{\partial V}{\partial t} - \gamma V + H(R,\nabla V) = 0, \tag{7}$$

with terminal condition

$$V(t_f,R) = \psi(R), \quad R \in SO(3),$$

and Hamiltonian function

$$H(R,P) := \min_{\Omega\in\mathfrak{so}(3)}\left\{\frac{1}{2}d^2_{SO(3)}(I,R) + \Psi_L(\Omega) + \langle P, R\Omega\rangle_R\right\}. \tag{8}$$

Lemma 2. The optimal control $\Omega^* \in \mathfrak{so}(3)$ which minimizes (8) is given by

$$\Omega^* = -\Sigma_L^{-1}(\mathbb{P}_a(R^T P)) \in \mathfrak{so}(3), \quad (9)$$

with Hamiltonian function

$$H(R,P) = \frac{1}{2}d^2_{SO(3)}(I,R) - \Psi_{\sigma(\rho(L)^{-1})}\left(\mathbb{P}_a(R^T P)\right).$$

Proof. The control input $\Omega \in \mathfrak{so}(3)$ is subject to the constraint $\Omega + \Omega^T = 0$. Thus, we must perform the following minimization

$$\min_{\Omega}\left(\frac{1}{2}d^2_{SO(3)}(I,R) + \Psi_L(\Omega) + \langle P, R\Omega\rangle_R + \frac{1}{2}\langle\langle\Lambda, \Omega^T + \Omega\rangle\rangle\right), \quad (10)$$

where $\Lambda = \Lambda^T$ is a Lagrange multiplier. The necessary condition for the optimization of equation (10) gives

$$R^T P + \Sigma_L(\Omega) + \Lambda = 0. \quad (11)$$

Since $\Sigma_L(\Omega) \in \mathfrak{so}(3)$, the above condition implies that the term $(R^T P + \Lambda)$ must be an element of the Lie algebra $\mathfrak{so}(3)$, *i.e.*, $(R^T P + \Lambda) \in \mathfrak{so}(3)$ or

$$(R^T P + \Lambda)^T + (R^T P + \Lambda) = 0.$$

We can uniquely solve for Λ to obtain

$$\Lambda = -\frac{1}{2}(R^T P + P^T R).$$

Substituting this expression in equation (11), yields (9).

Using the result of Lemma 2 in equation (7) leads to

$$\frac{\partial V}{\partial t} - \gamma V - \Psi_{\sigma(\rho(L)^{-1})}\left(R^T \nabla V\right) + \frac{1}{2}d^2_{SO(3)}(I,R) = 0. \quad (12)$$

Note that a solution to the above general PDE is *unique*, thus if one could find a value function V that verifies this PDE, the optimal control law is directly determined by (9) setting $P = \nabla V$.

An attempt to solve this problem using the Euclidean distance on SO(3) appeared in Saccon et al. [2010]. The weighted Euclidean distance on SO(3) is defined by

$$\frac{1}{2}d^2_{SO(3)}(I,R) = \frac{1}{2}||I - R||^2 = \mathrm{tr}(I - R).$$

In Saccon et al. [2010] the authors gave an "explicit" solution in the particular case of an infinite time-horizon with scalar penalty $L = \alpha I_{3\times 3}$. The solution to the optimal control problem was given by the feedback law

$$\Omega^* = -\frac{2}{\sqrt{\alpha}}\frac{\mathbb{P}_a(R)}{\sqrt{1 + \mathrm{tr}(R)}},$$

and the value function

$$V(R) = 4\sqrt{\alpha}(2 - \sqrt{1 + \mathrm{tr}(R)}), \quad (13)$$

which corresponds to the continuous (but non differentiable) viscosity solution of the HJB equation.

In the well known LQR problem on $\mathbb{R}^n$ (1), the solution to the HJB equation is the quadratic value function $V := x^T K x$, where K is solution of a Riccati equation. Unfortunately, using the Euclidean distance on $SO(3)$, the value function given in (13) does not have the same quadratic form as the cost $||I - R||^2$ (not an Euclidean distance), which shows the difficulty of solving the HJB equation using this type of distance for the general case of finite horizon problems, and/or arbitrary weighting matrices.

4. OPTIMAL SOLUTION USING THE GEODESIC DISTANCE ON SO(3)

A more geometrically meaningful metric on $SO(3)$ is known as the *geodesic* distance. It is defined as the length of the shortest path between two elements of the Lie group $SO(3)$. For two elements $R1, R_2 \in SO(3)$, the geodesic distance is given by [Huynh, 2009]

$$d(R_1, R_2) = ||\log(R_2^T R_1)||.$$

Using this metric, we can solve the following quadratic optimal control problems on $SO(3)$ where Riccati-like equations are derived to show the analogy with the linear case.

4.1 Infinite horizon optimal control problem on SO(3) with a discount rate

Consider the following optimal control problem on $SO(3)$

$$\min_{\Omega\in\mathfrak{so}(3)} J(\Omega) := \min_{\Omega\in\mathfrak{so}(3)} \frac{1}{2}\int_0^\infty \left\{||\log(R)||^2 + \alpha||\Omega||^2\right\} e^{-\gamma t} dt \quad (14)$$

subject to $\dot{R} = R\Omega$.

Proposition 1. The feedback law that minimizes the value of the cost $J(\Omega)$ is

$$\Omega^* = -\frac{k}{2\alpha}\log(R),$$

where the scalar gain k is given by

$$k = -\alpha\gamma + \sqrt{(\alpha\gamma)^2 + 2\alpha}.$$

Proof. The *time-invariant* value function $V(R)$ must satisfy the following Hamilton-Jacobi equation

$$-\gamma V(R) + H(R, \nabla V(R)) = 0,$$

with terminal condition

$$\lim_{t\to\infty} V(R) = 0,$$

and Hamiltonian function

$$H(R,P) := \min_{\Omega\in\mathfrak{so}(3)}\left\{\frac{1}{2}||\log(R)||^2 + \frac{\alpha}{2}||\Omega||^2 + \langle P, R\Omega\rangle_R\right\}.$$

A natural guess of the value function is

$$V(R) := \frac{k}{2}||\log(R)||^2. \quad (15)$$

To calculate the gradient of this function, we use the following result [Bullo and Lewis, 2005]

$$\frac{d}{dt}(\log(R)) = \mathcal{B}^+\Omega, \quad (16)$$

where the operator $\mathcal{B}^+$ is defined as

$$\mathcal{B}^+ := \mathrm{id}_{\mathfrak{so}(3)} + \frac{1}{2}\mathrm{ad}_{\log(R)} + \frac{1 - y(\theta)}{\theta^2}\mathrm{ad}^2_{\log(R)},$$

where θ is given by (5) and $y(\theta) = (\theta/2)\cot(\theta/2)$. The map $\mathrm{id}_{\mathfrak{so}(3)}$ represents the identity map on the Lie algebra $\mathfrak{so}(3)$ and $\mathrm{ad}^2_{\log(R)} := \mathrm{ad}_{\log(R)} \circ \mathrm{ad}_{\log(R)}$. For the subsequent development, we define also the operator

$$\mathcal{B}^- := \mathrm{id}_{\mathfrak{so}(3)} - \frac{1}{2}\mathrm{ad}_{\log(R)} + \frac{1 - \alpha(\theta)}{\theta^2}\mathrm{ad}^2_{\log(R)}.$$

The time derivative of (15) is

$$\dot{V} = k\langle\langle \frac{d}{dt}(\log(R)), \log(R)\rangle\rangle = k\langle\langle \mathcal{B}^{+}\Omega, \log(R)\rangle\rangle = k\langle\langle \Omega, \mathcal{B}^{-}\log(R)\rangle\rangle.$$

where we have used the fact that

$$\langle\langle \mathrm{ad}_{\log(R)}\Omega, \log(R)\rangle\rangle = -\langle\langle \Omega, \mathrm{ad}_{\log(R)}\log(R)\rangle\rangle$$
$$\langle\langle \mathrm{ad}^2_{\log(R)}\Omega, \log(R)\rangle\rangle = \langle\langle \Omega, \mathrm{ad}^2_{\log(R)}\log(R)\rangle\rangle.$$

However, since $\mathrm{ad}_{\log(R)}\log(R) = 0$, one has $\mathcal{B}^{-}\log(R) = \log(R)$. Therefore,

$$\dot{V} = k\langle\langle \Omega, \log(R)\rangle\rangle. \tag{17}$$

Using the fact that

$$\dot{V} = \langle \dot{R}, \nabla V\rangle_R = \langle R\Omega, \nabla V\rangle_R = \langle\langle \Omega, R^T\nabla V\rangle\rangle,$$

one has

$$\nabla V = kR\log(R). \tag{18}$$

Consequently, using the result of Lemma 2 , the corresponding HJB equation, reads

$$-\frac{\gamma k}{2}||\log(R)||^2 - \frac{k^2}{4\alpha}||\log(R)||^2 + \frac{1}{2}||\log(R)||^2 = 0,$$

which implies

$$-\gamma k - \frac{k^2}{2\alpha} + 1 = 0. \tag{19}$$

The above algebraic Riccati equation can be explicitly solved for the positive scalar k to obtain

$$k = -\alpha\gamma + \sqrt{(\alpha\gamma)^2 + 2\alpha}.$$

Using the result of Lemma 2, we have

$$\Omega^* = -\frac{k}{2\alpha}\log(R).$$

Remark 1. A solution to this infinite horizon optimal control problem with a "discount rate" was not possible to obtain using the traditional Euclidean distance, since it is not obvious to get an algebraic Riccati equation as in (19).

4.2 Finite time horizon optimal control problem on SO(3)

Consider the following optimal control problem:

$$\min_{\Omega\in\mathfrak{so}(3)} J(t_0, R_0, \Omega) := \min_{\Omega\in\mathfrak{so}(3)} \frac{1}{2}\int_{t_0}^{t_f} \left\{||\log(R)||^2 + \alpha||\Omega||^2\right\} dt + \frac{k_f}{2}||\log(R(t_f))||^2$$

subject to $\dot{R} = R\Omega$.

Proposition 2. The feedback law that minimizes the value of the cost $J(t_0, R_0, \Omega)$ is

$$\Omega^*(t, R) = -\frac{k(t)}{2\alpha}\log(R), \tag{20}$$

where the time varying scalar gain $k(t)$ is given by

$$\begin{cases} \dot{k}(t) - \dfrac{k(t)^2}{2\alpha} + 1 = 0 \\ k(t_f) = k_f \end{cases} \tag{21}$$

4.3 Arbitrary weighting matrices

Consider the general finite time-horizon optimal control problem on $SO(3)$ with arbitrary positive definite weighting matrices

$$\min_{\Omega\in\mathfrak{so}(3)} J(t_0, R_0, \Omega) := \min_{\Omega\in\mathfrak{so}(3)} \int_{t_0}^{t_f} \mathrm{e}^{-\gamma t}\left\{\Psi_M(\log(R)) + \Psi_L(\Omega)\right\} dt + e^{-\gamma t_f}\Psi_F(\log(R(t_f))) \tag{22}$$

subject to $\dot{R} = R\Omega$. A solution to this problem for small rotations is stated in the following theorem.

Theorem 1. For small rotations around $R = I$, the solution to the optimal control problem (22) on SO(3) is given by

$$\Omega^* = -\Sigma_L^{-1}\Sigma_K(\log(R)),$$

where the positive definite matrix K is solution of the following dynamic Riccati equation

$$\begin{cases} \rho(\dot{K}) - \gamma\rho(K) - \rho(K)\rho(L)^{-1}\rho(K) + \rho(M) = 0 \\ K(t_f) = F \end{cases}$$

where the map $\rho(.)$ is defined as

$$\rho(A) := tr(A)I - A, \quad \forall A \in \mathbb{R}^{3\times 3}.$$

4.4 Inverse optimal control problem

The inverse optimal control approach consists in the design of control laws that are optimal with respect to a meaningful cost functional without solving a Hamilton-Jacobi equation [Freeman and Kokotovic, 1996, Sepulchre et al., 1997]. We briefly formulate the inverse optimality control problem on the Lie group $SO(3)$. Let W be a positive definite potential function on $SO(3)$, then we have

$$\dot{W} = \langle \nabla W, R\Omega\rangle_R = \langle\langle R^T\nabla W, \Omega\rangle\rangle. \tag{23}$$

If we design a kinematic state feedback control law as

$$\Omega := \kappa(R) = -\Sigma_L^{-1}\left(R^T\nabla W\right), \tag{24}$$

for some positive definite matrix $L \in \mathbb{R}^{3\times 3}$, we guarantee that $\dot{W} \le 0$. Generally, it is shown that $\dot{W} \to 0$ and $R \to I$ (almost globally) as t goes to infinity.
We have the following result.

Proposition 3. The control law

$$\Omega = -\beta\Sigma_L^{-1}\left(R^T\nabla W\right), \quad \beta > 0$$

is optimal with respect to the cost

$$J = \int_0^\infty \mathcal{C}(R, \Omega)dt,$$

where $\mathcal{C}(R, \Omega) = \beta^2\Psi_{\sigma(\rho(L))^{-1}}\left(R^T\nabla W\right) + \Psi_L(\Omega)$.

In particular, let us consider the geodesic potential function on $SO(3)$

$$W = \frac{1}{2}||\log(R)||^2,$$

whose gradient is given by

$$\nabla W = R\log(R).$$

Consequently, taking the kinematic control law as

$$\Omega = -\Sigma_L^{-1}(\log(R))) = -\Sigma_M(\log(R))), \tag{25}$$

where $M = \sigma(\rho(L)^{-1})$, yields

$$\dot{W} = -\Psi_M(\log(R)) \le -2\lambda^M_{\min}W,$$

where $\lambda^M_{\min}$ is the smallest eigenvalue of the positive definite matrix $\rho(M) = \rho(L)^{-1}$. Clearly, one can conclude that the feedback law (25) exponentially stabilizes the system at $R = I$ from all initial conditions with $\mathrm{tr}(R) \neq -1$.

Proposition 3 suggests that the control law

$$\Omega = -\beta \Sigma_M(\log(R)),$$

is optimal with respect to the cost

$$J = \int_0^\infty \left\{ \beta^2 \Psi_M \left(\log(R)\right) + \Psi_L(\Omega) \right\} dt.$$

Note that this optimal solution is *global* in contrast to the local solution of the precedent section.

5. CONCLUSION

The optimal kinematic control problem on the Lie group SO(3) has been investigated. A solution to this problem has been carried out using a natural geodesic distance on SO(3). An interesting analogy of the obtained optimal solution on SO(3) with the well known results for linear systems has been established, where the optimal kinematic control law is a state feedback (using the exponential coordinates) with a gain depending on the solution of a Riccati-like equation.

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