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Model reference adaptive iterative learning control for linear systems

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SUMMARY

In this paper, we propose a model reference adaptive control (MRAC) strategy for continuous-time single-input single-output (SISO) linear time-invariant (LTI) systems with unknown parameters, performing repetitive tasks. This is achieved through the introduction of a discrete-type parametric adaptation law in the 'iteration domain', which is directly obtained from the continuous-time parametric adaptation law used in standard MRAC schemes. In fact, at the first iteration, we apply a standard MRAC to the system under consideration, while for the subsequent iterations, the parameters are appropriately updated along the iteration-axis, in order to enhance the tracking performance from iteration to iteration. This approach is referred to as the model reference adaptive iterative learning control (MRAILC). In the case of systems with relative degree one, we obtain a pointwise convergence of the tracking error to zero, over the whole finite time interval, when the number of iterations tends to infinity. In the general case, i.e. systems with arbitrary relative degree, we show that the tracking error converges to a prescribed small domain around zero, over the whole finite time interval, when the number of iterations tends to infinity. It is worth noting that this approach allows: (1) to extend existing MRAC schemes, in a straightforward manner, to repetitive systems; (2) to avoid the use of the output time derivatives, which are generally required in traditional iterative learning control (ILC) strategies dealing with systems with high relative degree; (3) to handle systems with multiple tracking objectives (i.e. the desired trajectory can be iteration-varying). Finally, simulation results are carried out to support the theoretical development. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Adaptive control is one of the most popular control techniques that has been fascinating the automatic control community for several years [1, 2]. In the standard adaptive control

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framework, the parametric adaptation rule is generally an integration along the time-axis, which is commonly designed using the Lyapunov method in order to achieve asymptotic tracking. Hence, the tracking objective is achieved along an infinite time interval, and a transient tracking error will always be present. Model reference adaptive control (MRAC) is among the famous adaptive techniques that have been around for more than three decades. The major problem that one can attribute to this technique is the bad transient performance. To overcome this drawback, other alternatives, such as the backstepping approach [3], have been proposed in the literature. However, the benefit brought by those techniques in terms of transient improvement, is often eclipsed by the control law implementation complexity. On the other hand, in practical applications, the designed controller can be applied more than once to the plant under consideration, over a finite time interval. In this case, one can benefit from the information collected at the previous operations in order to enhance the transient performance for the subsequent operations. This technique is known as ILC [4]. Most of the existing ILC schemes in the literature are based upon the contraction mapping technique and require a certain *a priori* knowledge of the system parameters, the use of the time-weighted norm in the convergence proofs as well as the use of the output time derivatives for systems with high relative degree [5–7]. Recently, a growing interest has been directed towards the energy-based approach which takes its essence from the Lyapunov theory [8–13]. This approach uses an elegant and powerful framework for stability and convergence analysis which has shown a real effectiveness in handling systems with time-varying parameters and multiple tracking objectives (i.e. the desired trajectory can be modified from iteration to iteration). On the other hand, in Reference [14], an adaptive approach to ILC has been proposed, where the iterative parametric adjustment is performed on the initial conditions of the continuous-time integral-type adaptation law. In other words, a standard adaptive controller is used and the parameter estimates are initialized with their final values obtained at the preceding iteration. Therefore, this technique inherits the limitations associated to standard adaptive control such as the requirement of the unknown system parameters to be constant.

In this paper,[‡] we provide an extension of standard MRAC schemes to single-input single-output linear time-invariant systems performing repetitive tasks. The proposed model reference adaptive iterative learning control strategy achieves a global asymptotic tracking along the time horizon at the first iteration, and a pointwise convergence of the tracking error to zero (in the case of systems with relative degree one) or to a prescribed small domain around zero (for systems with higher relative degree), over the whole finite time interval, when the number of iterations tends to infinity. In fact, at the first iteration, we use a continuous-time integral-type parametric adaptation law, while for the subsequent iterations, we use a discrete-type parametric adaptation law along the iteration-axis. The proof of convergence is based upon the use of a Lyapunov-like sequence which is shown to be monotonically decreasing along the iterative process. Basically, the role of the discrete-type parametric adaptation law is to refine the transient response from iteration to iteration in order to achieve an accurate tracking over a finite time interval. In contrast to existing contraction mapping-based ILC schemes, the proposed control strategy does not require the use of the output time derivatives for systems with a high relative degree, and is able to handle varying tracking objectives throughout the iterative process. Finally, it is worth noting that in the recent paper [16], an elegant output-based

[‡]A preliminary version of this work has been presented in Reference [15].

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adaptive ILC has been proposed for uncertain linear systems using the MRAC approach. However, the main difference between our approach and the one proposed in Reference [16] is related to the following facts: (1) We extend the well-known SPR-Lyapunov approach, used in adaptive control, for repetitive systems; (2) The main objective of our paper is to show that it is possible to extend, in a straightforward manner, standard MRAC schemes to repetitive systems; therefore, for the sake of presentation simplicity, we considered the MRAC versions proposed in References [1, 2] in their 'simplest form'. Nevertheless, the proposed framework can be easily generalized to more complex MRAC schemes dealing with the issues of robustness, disturbances, measurement noise, etc. (3) In Reference [10], the parameter estimates at the first iteration are arbitrarily set, while in our approach, they are obtained using a continuous-time integral-type parametric adaptation rule, and hence a certain level of performance is achieved already at the first iteration; (4) The bounds of the system parameters involved in the projection mechanism used in Reference [16] are not required in our approach.

2. PROBLEM FORMULATION

In this paper we consider SISO-LTI systems described by

$$y_k(t) = G_p(s)[u_k(t)] = k_p \frac{Z_p(s)}{R_p(s)}[u_k(t)] \tag{1}$$

and operated repeatedly over a finite time interval $[0, T]$. The non-negative integer $k \in \mathbb{Z}_+$ denotes the iteration or trial number. The desired trajectory $y_d(t)$ is given by a reference model as follows:

$$y_d(t) = G_m(s)[r_f(t)] = k_m \frac{Z_m(s)}{R_m(s)}[r_f(t)] \tag{2}$$

where $r_f(t)$ is a bounded reference input.

Assuming that the system parameters are unknown (except the sign of the high-frequency-gain k_p), our objective is to design an adaptive iterative learning controller guaranteeing the boundedness of the tracking error $\forall t \in [0, T]$ and $\forall k \in \mathbb{Z}_+$, and its convergence to a small neighbourhood of zero, over the whole finite time interval $[0, T]$, when k tends to infinity. To this end, we will assume that $y_d(0) = y_k(0)$ and without any loss of generality we will assume that $y_d(0) = y_k(0) = 0$. Throughout this paper, we will use the $\mathscr{L}_{pe}$ norm defined as follows:

$$\|x(t)\|_{pe} \triangleq \begin{cases} \left(\int_0^t \|x(\tau)\|^p \, \mathrm{d}\tau\right)^{1/p} & \text{if } p \in [0, \infty) \\ \sup_{0 \leqslant \tau \leqslant t} \|x(\tau)\| & \text{if } p = \infty \end{cases}$$

where $\|x\|$ denotes any norm of x, and t belongs to the finite interval $[0, T]$. We say that $x \in \mathscr{L}_{pe}$ when $\|x\|_{pe}$ exists (i.e. when $\|x\|_{pe}$ is finite). We will also make the following classical assumptions related to the MRAC technique:

- (B1) Z_p is a monic Hurwitz polynomial of degree m_p.
- (B2) An upper bound n of the degree n_p of $R_p(s)$ is available.
- (B3) The relative degree $r = n_p - m_p$ of G_p is known.
- (B4) The sign of the high frequency gain k_p is known.

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(B5) Z_m and R_m are monic Hurwitz polynomial of degree m_m and n_m, respectively, with $n_m \leqslant n$.
(B6) The relative degree $r_m = n_m - m_m$ of G_m is the same as that of G_p.

Note that the minimum-phase requirement on the plant stated in assumption (B1) is not necessary in our developments since our ILC operates over a finite-time interval. Therefore, the boundedness of the states is guaranteed over any finite time interval since a finite escape-time is not possible for the class of linear systems considered here.

3. PRELIMINARIES

Let us define $\Lambda(s) = \Lambda_0(s)Z_m(s)$, which is a monic Hurwitz polynomial of degree $n-1$. Define also $\alpha(s)$ as follows:

$$\alpha(s) = \begin{cases} [s^{n-2}, s^{n-1}, \ldots, s, 1]^{\mathrm{T}} & \text{for } n \geqslant 2 \\ 0 & \text{for } n = 1 \end{cases}$$

As shown in References [1, 2], there exists a set of parameters $c_0^* \in \mathbb{R}$, $\theta_3^* \in \mathbb{R}$, $\theta_1^* \in \mathbb{R}^{n-1}$ and $\theta_2^* \in \mathbb{R}^{n-1}$ such that the following control law:

$$u_k = \theta^{*\mathrm{T}} \Omega_k$$

where

$$\theta^* = [\theta_1^{*\mathrm{T}}, \theta_2^{*\mathrm{T}}, \theta_3^*, c_0^{*\mathrm{T}}], \quad \Omega_k = [w_{1,k}^{\mathrm{T}}, w_{2,k}^{\mathrm{T}}, y_k, r_f]^{\mathrm{T}}, \quad w_{1,k} = \frac{\alpha(s)}{\Lambda(s)}[u_k] \quad \text{and} \quad w_{2,k} = \frac{\alpha(s)}{\Lambda(s)}[y_k]$$

leads to

$$y_k = G_m(s)[r_f] = k_m \frac{Z_m}{R_m}[r_f]$$

The parameters can be obtained from the following relationships:

$$c_0^* = \frac{k_m}{k_p}$$

$$(\Lambda - \theta_1^{*\mathrm{T}}\alpha)R_p - k_p Z_p(\theta_2^{*\mathrm{T}}\alpha + \theta_3^*\Lambda) = Z_p \Lambda_0 R_m$$

The signals $w_{1,k}$ and $w_{2,k}$ are the outputs of the following systems:

$$\dot{w}_{1,k} = F w_{1,k} + g u_k \quad w_{1,k}(0) = 0$$

$$\dot{w}_{2,k} = F w_{2,k} + g y_k \quad w_{2,k}(0) = 0 \tag{3}$$

where (F, g) is a state-space realization of $\alpha(s)/\Lambda(s)$.

The state-space representation of the overall closed-loop system is given by the following non-minimal realization:

$$\dot{Y}_{c,k} = A_c Y_{c,k} + B_c c_0^* r_f$$

$$y_k = C_c Y_{c,k} \tag{4}$$

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with $Y_{c,k} = [x_k^T, w_{1,k}^T, w_{2,k}^T]^T \in \mathbb{R}^{n_p+2n-2}$, where x_k denotes the state vector associated with system (1), and

$$A_c = \begin{bmatrix} A + B\theta_3^* C & B\theta_1^{*T} & B\theta_2^{*T} \\ g\theta_3^* C & F + g\theta_1^{*T} & g\theta_2^{*T} \\ gC & 0 & F \end{bmatrix}, \quad B_c = \begin{bmatrix} B \\ g \\ 0 \end{bmatrix}, \quad C_c = [C, \ 0, \ 0] \tag{5}$$

Hence, the transfer function from r_f to y_k is given by

$$\frac{y_k}{r_f} = G_m(s) = C_c(sI - A_c)^{-1} B_c c_0^*$$

Therefore, the reference model can also be described by

$$\dot{Y}_m = A_c Y_m + B_c c_0^* r_f$$

$$y_d = C_c Y_m$$

Note that A_c is a stable matrix, since $det(sI - A_c) = \Lambda(s) Z_p(s) \Lambda_0(s) R_m(s)$.

Let $e_k = Y_{c,k} - Y_m$ be the state error and $e_{1,k} = y_k - y_d$ be the output tracking error. It follows that

$$\dot{e}_k = A_c e_k$$

$$e_{1,k} = C_c e_k \tag{6}$$

which shows that the tracking error converges exponentially to zero.

Since the system parameters are unknown, the vector θ^* cannot be obtained and hence, the control law $u_k(t) = \theta^{*T}\Omega_k(t)$ cannot be applied. In this case, the MRAC technique consists of applying a control law of the form $u_k(t) = \theta_k^T(t)\Omega_k(t)$, where $\theta_k(t)$ is generated by an appropriate adaptive law.

In our approach, at the first iteration, i.e. for $k = 0$, the vector $\theta_0(t)$ is generated by a continuous-time integral-type adaptive law as in the usual MRAC framework, whereas for $k \geqslant 1$, the vector $\theta_k(t)$ is generated by a discrete integral-type adaptive law (iterative law along the iteration-axis).

4. MRAILC FOR SYSTEMS WITH RELATIVE DEGREE ONE

The following lemma is instrumental in our MRAILC design for systems with relative degree one.

4.1. Iterative-SPR-Lyapunov lemma

In this section, we propose an extended version of the positive real (SPR)-Lyapunov approach [1, 17].

Lemma 1

Let the signals $\bar{e}_k(t)$ and $\gamma\Psi_k^T(t)v_k(t)$ be related by a strictly positive real (SPR) transfer function $H(s)$ as follows:

$$\bar{e}_k(t) = H(s)[\gamma\Psi_k^T(t)v_k(t)] \tag{7}$$

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where t belongs to the finite time interval $[0, T]$, $\bar{e}_k(t) \in \mathbb{R}$, γ is an unknown constant with known sign, $v_k(t) \in \mathbb{R}^m$ is a measurable vector. The vector $\Psi_k(t) \in \mathbb{R}^m$ is generated by

$$\Psi_k(t) = \Psi_{k-1}(t) - \Gamma v_k(t)\bar{e}_k(t)\,\mathrm{sgn}(\gamma) \quad \text{for } k \geqslant 1 \tag{8}$$

and

$$\dot{\Psi}_0(t) = -\Gamma v_0(t)\bar{e}_0(t)\,\mathrm{sgn}(\gamma) \tag{9}$$

where $\Gamma \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix. Then

- The state vector $\bar{X}_k \in \mathscr{L}_{\infty e}$, $\bar{e}_k(t) \in \mathscr{L}_{\infty e}$ and $\Psi_k(t) \in \mathscr{L}_{2e}$, for all $k \in \mathbb{Z}_+$.
- $\lim_{k\to\infty} \bar{e}_k(t) = 0$, for all $t \in [0, T]$.

Proof
Let the state-space representation of (7) be

$$\begin{aligned} \dot{\bar{X}}_k &= \bar{A}\bar{X}_k + \bar{B}(\gamma\Psi_k^{\mathrm{T}}(t)v_k(t)), \quad \bar{X}_k(0) = 0 \\ \bar{e}_k &= \bar{C}\bar{X}_k \end{aligned} \tag{10}$$

Since $H(s)$ is SPR then, from Meyer–Kalman–Yakubovich (MKY) lemma [1, 17], for any given symmetric positive definite matrix L there exist a symmetric positive definite matrix P, a vector q and a strictly positive scalar v such that

$$\begin{aligned} \bar{A}^{\mathrm{T}}P + P\bar{A} &= -qq^{\mathrm{T}} - vL \\ P\bar{B} &= \bar{C}^{\mathrm{T}} \end{aligned} \tag{11}$$

Now, let us consider the following Lyapunov-like functional candidate:

$$W_k(\bar{X}_k, \Psi_k) = V_k(\bar{X}_k) + \frac{|\gamma|}{2}\int_0^t \Psi_k^{\mathrm{T}}(\tau)\Gamma^{-1}\Psi_k(\tau)\,\mathrm{d}\tau \tag{12}$$

with

$$V_k(\bar{X}_k) = \tfrac{1}{2}\bar{X}_k^{\mathrm{T}}P\bar{X}_k \tag{13}$$

which can be written, in view of (10) and (11), as follows:

$$\begin{aligned} V_k(\bar{X}_k(t)) &= V_k(\bar{X}_k(0)) + \int_0^t \dot{V}_k(\bar{X}_k(\tau))\,\mathrm{d}\tau \\ &= V_k(\bar{X}_k(0)) - \frac{1}{2}\int_0^t (\bar{X}_k^{\mathrm{T}}(qq^{\mathrm{T}} + vL)\bar{X}_k - 2\bar{X}_k^{\mathrm{T}}P\bar{B}(\gamma\Psi_k^{\mathrm{T}}(t)v_k))\,\mathrm{d}\tau \\ &= -\frac{1}{2}\int_0^t (\bar{X}_k^{\mathrm{T}}(qq^{\mathrm{T}} + vL)\bar{X}_k - 2\bar{e}_k(\gamma\Psi_k^{\mathrm{T}}(t)v_k))\,\mathrm{d}\tau \end{aligned} \tag{14}$$

Now, the difference of the Lyapunov-like functional (12) is given by

$$\Delta W_k = W_k - W_{k-1} = V_k - V_{k-1} - \frac{|\gamma|}{2}\int_0^t (\bar{\Psi}_k^{\mathrm{T}}\Gamma^{-1}\bar{\Psi}_k - 2\bar{\Psi}_k^{\mathrm{T}}\Gamma^{-1}\Psi_k)\,\mathrm{d}\tau \tag{15}$$

where $\bar{\Psi}_k = \Psi_k - \Psi_{k-1}$. Now, in view of (8) and (14), Equation (15) leads to

$$\Delta W_k = -V_{k-1} - \frac{|\gamma|}{2}\int_0^t \bar{\Psi}_k^{\mathrm{T}}\Gamma^{-1}\bar{\Psi}_k \,\mathrm{d}\tau - \frac{1}{2}\int_0^t \bar{X}_k^{\mathrm{T}}(qq^{\mathrm{T}} + \nu L)\bar{X}_k \,\mathrm{d}\tau \leqslant 0 \tag{16}$$

Hence $W_k(t)$ is non-increasing and consequently $\bar{X}_k(t)$, $\int_0^t \Psi_k^{\mathrm{T}}(\tau)\Gamma^{-1}\Psi_k(\tau)\,\mathrm{d}\tau$ and $\bar{e}_k(t)$ are bounded if $W_0(t)$ is bounded.

Now, to prove the boundedness of $W_0(t)$ let us consider the following Lyapunov function:

$$S_0(\bar{X}_0, \Psi_0) = \frac{1}{2}\bar{X}_0^{\mathrm{T}} P \bar{X}_0 + \frac{|\gamma|}{2}\Psi_0^{\mathrm{T}}\Gamma^{-1}\Psi_0 \tag{17}$$

whose time derivative in view of (9), (10) and (11) is given by

$$\dot{S}_0 = -\tfrac{1}{2}\bar{X}_0^{\mathrm{T}}(qq^{\mathrm{T}} + \nu L)\bar{X}_0 \tag{18}$$

which means that $\bar{X}_0(t)$ and $\Psi_0(t)$ are globally bounded. Hence, $W_0(t)$ is bounded over the finite time interval $[0, T]$.

To show the convergence of $\bar{e}_k(t)$ to zero when k tends to infinity, let us rewrite W_k as follows:

$$W_k = W_0 + \sum_{j=1}^{j=k} \Delta W_j \leqslant W_0 - \sum_{j=1}^{j=k} V_{j-1}$$

$$\leqslant W_0 - \frac{1}{2}\sum_{j=1}^{j=k} \bar{X}_{j-1}^{\mathrm{T}} P \bar{X}_{j-1}$$

which leads to

$$\sum_{j=1}^{j=k} \bar{X}_{j-1}^{\mathrm{T}}(t) P \bar{X}_{j-1}(t) \leqslant 2(W_0(t) - W_k(t)) \leqslant 2W_0(t) \tag{19}$$

Since $W_0(t)$ and $\bar{X}_k(t)$ are bounded for all $k \in \mathbb{Z}_+$ and $t \in [0, T]$, one can conclude that $\lim_{k\to\infty} \bar{X}_k(t) = 0$ and consequently $\lim_{k\to\infty} \bar{e}_k(t) = 0$, $\forall t \in [0, T]$. □

Remark 1

Note that by virtue of Barbalat lemma, and under the assumption that $v_0(t)$ is bounded for all $t \in \mathbb{R}^+$, one can easily show that $\lim_{t\to\infty} \bar{e}_0(t) = 0$.

Remark 2

If $v_k(t) \in \mathscr{L}_{\infty e}$ for any finite non-negative integer k, one can show that $\Psi_k(t) \in \mathscr{L}_{\infty e}$ for any finite non-negative integer k.

4.2. MRAILC design

For systems with relative degree $r = 1$, the design of an MRAILC is straightforward from Lemma1 as stated in the following theorem.

Theorem 1

Assume that (B1–B6) are satisfied and $G_m(s)$ is SPR. Consider system (1), with a relative degree $r = 1$, under the following control law:

$$u_k(t) = \theta_k^{\mathrm{T}}(t)\Omega_k(t) \quad \text{for } k \geqslant 0 \tag{20}$$

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where at the first iteration, i.e. $k = 0$, the parametric adaption law is given by

$$\dot{\theta}_0(t) = -\Gamma\Omega_0(t)e_{1,0}(t)\,\text{sgn}(\rho^*) \tag{21}$$

and for $k \geqslant 1$, the parametric adaption law is given by

$$\theta_k(t) = \theta_{k-1}(t) - \Gamma\Omega_k(t)e_{1,k}(t)\,\text{sgn}(\rho^*) \tag{22}$$

where $\rho^* = k_p/k_m$ and $\Gamma \in \mathbb{R}^{2n\times 2n}$ is a symmetric positive definite matrix. Then,

- The state vector $Y_{c,k} \in \mathscr{L}_{\infty e}$, $e_{1,k}(t) = (y_k(t) - y_d(t)) \in \mathscr{L}_{\infty e}$ and $\theta_k(t) \in \mathscr{L}_{2e}$, for all $k \in \mathbb{Z}_+$.
- $\lim_{k\to\infty} e_{1,k}(t) = 0$, for all $t \in [0, T]$.

Proof

Since (6) is obtained with $u_k(t) = \theta^{*\text{T}}\Omega_k(t)$, one has

$$\begin{aligned} \dot{e}_k &= A_c e_k + B_c(u_k - \theta^{*\text{T}}\Omega_k) \\ e_{1,k} &= C_c e_k \end{aligned} \tag{23}$$

which under the control law (20) becomes

$$\begin{aligned} \dot{e}_k &= A_c e_k + B_c\tilde{\theta}_k^{\text{T}}\Omega_k \\ e_{1,k} &= C_c e_k \end{aligned} \tag{24}$$

with $\tilde{\theta}_k(t) = \theta_k(t) - \theta^*$. Since $C_c(sI - A_c)^{-1}B_c c_0^* = G_m(s)$, system (24) leads to

$$e_{1,k} = G_m(s)[\rho^*\tilde{\theta}_k^{\text{T}}(t)\Omega_k(t)] \tag{25}$$

where $\rho^* = 1/c_0^*$. Finally, under the adaptive laws (21) and (22), the result follows directly from Lemma 1. □

Remark 3

Since $r_f, e_{1,k}, Y_{c,k} \in \mathscr{L}_{\infty e}$ $\forall k \in \mathbb{Z}_+$, one can conclude that $\Omega_k \in \mathscr{L}_{\infty e}$ $\forall k \in \mathbb{Z}_+$. Hence, one can show that $\theta_k(t) \in \mathscr{L}_{\infty e}$ for any finite non-negative integer k. Consequently, $u_k(t) \in \mathscr{L}_{\infty e}$ for any finite non-negative integer k.

Remark 4

Note that, for $k = 0$, the control scheme proposed in Theorem 1 is nothing else but a standard MRAC. It turns out that the second term of the right-hand side of the discrete-type adaptation law (22) is similar to the right-hand side of the continuous-time adaptation law (21). This is due to the fact that the Lyapunov function (17) used to design the standard MRAC is extended to the repetitive case by substituting the quadratic term on the parametric error by its integral. For systems with a relative degree $r > 1$ direct application of the Iterative-SPR-Lyapunov lemma is not possible. Nevertheless, it is possible to obtain MRAILC schemes, in a straightforward manner, from the standard MRAC algorithms dealing with higher relative degrees (see, for instance, References [1, 2] and references therein), by associating to each continuous-time integral-type adaption

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law a discrete integral-type adaption law with saturation, along the iteration-axis as shown in the next section.

5. MRAILC FOR SYSTEMS WITH RELATIVE DEGREE $r \geqslant 1$

In this section, we propose a MRAILC scheme for systems with an arbitrary relative degree $r \geqslant 1$. Our result is based on the extension of the MRAC schemes proposed in References [1, 2].

Theorem 2

Assume that (B1–B6) are satisfied. Consider system (1), with a relative degree $r \geqslant 1$, under the following control law over $[0, T]$:

$$u_k(t) = \theta_k^{\mathrm{T}}(t)\Omega_k(t) \quad \text{for } k \geqslant 0 \tag{26}$$

where at the first iteration, i.e. $k = 0$, we use

$$\dot{\theta}_0(t) = -\Gamma \varepsilon_0(t)\phi_0(t)\,\mathrm{sgn}(\rho^*) \tag{27}$$

$$\dot{\rho}_0(t) = \gamma \varepsilon_0(t)\xi_0(t) \tag{28}$$

and for $k \geqslant 1$, we use

$$\theta_k(t) = \begin{cases} \theta_{k-1}(t) - \Gamma \varepsilon_k(t)\phi_k(t)\,\mathrm{sgn}(\rho^*) & \text{if } \sup_{t\in[0,T]} |\varepsilon_{k-1}(t)| > \sigma \\ \theta_{k-1}(T) & \text{otherwise} \end{cases} \tag{29}$$

$$\rho_k(t) = \begin{cases} \rho_{k-1}(t) + \gamma \varepsilon_k(t)\xi_k(t) & \text{if } \sup_{t\in[0,T]} |\varepsilon_{k-1}(t)| > \sigma \\ \rho_{k-1}(T) & \text{otherwise} \end{cases} \tag{30}$$

where $\rho^* = k_p/k_m$, $\Gamma \in \mathbb{R}^{2n\times 2n}$ is a symmetric positive definite matrix and γ is a positive parameter. The signals ϕ_k, ε_k and ξ_k are evaluated for all $k \in \mathbb{Z}_+$ as follows:

$$\varepsilon_k = \frac{e_{1,k} - \hat{e}_{1,k}}{m_k^2}$$

$$\hat{e}_{1,k} = \rho_k \xi_k$$

$$\xi_k = \bar{u}_k - \theta_k^{\mathrm{T}}\phi_k$$

$$\phi_k = G_m(s)[\Omega_k]$$

$$\bar{u}_k = G_m(s)[u_k]$$

$$m_k^2 = \begin{cases} 1 + \bar{u}_k^2 + \phi_k^{\mathrm{T}}\phi_k \quad \text{or} \quad 1 + \bar{\phi}_k^{\mathrm{T}}\bar{\phi}_k & \text{for } k = 0 \\ \kappa \quad \text{for } k \geqslant 1 \end{cases} \tag{31}$$

where κ is a positive parameter, $\bar{\phi}_k = G_m(s)[\bar{\Omega}_k]$, with $\bar{\Omega}_k = [w_{1,k}^{\mathrm{T}}, w_{2,k}^{\mathrm{T}}, y_k]^{\mathrm{T}}$.

Then, all signals are bounded $\forall k \in \mathbb{Z}_+$, $\forall t \in [0, T]$, and $\lim_{k\to\infty} |e_{1,k}(t)| \leqslant \kappa\sigma$, $\forall t \in [0, T]$.

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Proof
First, let us assume that $\sup_{t\in[0,T]}|\varepsilon_{k-1}(t)|>\sigma$ and let us consider the following Lyapunov-like function for $k\geqslant 1$:

$$W_k=\frac{1}{2\gamma}\tilde{\rho}_k^2+\frac{|\rho^*|}{2}\tilde{\theta}_k^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}_k \tag{32}$$

where $\tilde{\rho}_k=\rho_k-\rho^*$ and $\tilde{\theta}_k=\theta_k-\theta^*$. Using (26) and (29)–(31), and the fact that $e_{1,k}=\rho^*G_m[\tilde{\theta}_k^{\mathrm{T}}\Omega_k]=\rho^*(\bar{u}_k-\theta_k^{*\mathrm{T}}\phi_k)$, we have

$$\begin{aligned}\Delta W_k(t)=W_k(t)-W_{k-1}(t)&=-\frac{1}{2\gamma}\bar{\rho}_k^2-\frac{|\rho^*|}{2}\bar{\theta}_k^{\mathrm{T}}\Gamma^{-1}\bar{\theta}_k+\frac{1}{\gamma}\bar{\rho}_k\tilde{\rho}_k+|\rho^*|\bar{\theta}_k^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}_k\\&=-\frac{1}{2\gamma}\bar{\rho}_k^2(t)-\frac{|\rho^*|}{2}\bar{\theta}_k(t)^{\mathrm{T}}\Gamma^{-1}\bar{\theta}_k(t)-m_k^2\varepsilon_k^2(t)\leqslant 0\end{aligned} \tag{33}$$

with $\bar{\rho}_k(t)=\rho_k(t)-\rho_{k-1}(t)$ and $\bar{\theta}_k(t)=\theta_k(t)-\theta_{k-1}(t)$. Note that $m_k^2=\kappa>0$ since $k\geqslant 1$. It is clear that $W_k(t)$ is non-increasing, and hence bounded if $W_0(t)$ is bounded over $[0,T]$. Since, for $k=0$, all signals are bounded,[§] one can conclude that $W_0(t)$ is bounded for all $t\in[0,T]$, and hence $W_k(t)$ is bounded $\forall k\in\mathbb{Z}_+$, $\forall t\in[0,T]$. This implies that $\rho_k,\theta_k\in\mathscr{L}_{\infty e}$, $\forall k\in\mathbb{Z}_+$, $\forall t\in[0,T]$.

Now, one can show that

$$\begin{aligned}W_k&=W_0+\sum_{j=1}^{k}\Delta W_j\\&=W_0-\frac{1}{2\gamma}\sum_{j=1}^{k}\bar{\rho}_j^2-\frac{|\rho^*|}{2}\sum_{j=1}^{k}\bar{\theta}_j^{\mathrm{T}}\Gamma^{-1}\bar{\theta}_j-\sum_{j=1}^{k}\kappa\varepsilon_j^2\end{aligned} \tag{34}$$

thus,

$$\frac{1}{2\gamma}\sum_{j=1}^{k}\bar{\rho}_j^2+\frac{|\rho^*|}{2}\sum_{j=1}^{k}\bar{\theta}_j^{\mathrm{T}}\Gamma^{-1}\bar{\theta}_j+\sum_{j=1}^{k}\kappa\varepsilon_j^2=W_0-W_k\leqslant W_0 \tag{35}$$

Therefore, one can conclude that $\bar{\rho}_k,\bar{\theta}_k,\varepsilon_k\in\mathscr{L}_{\infty e}$, which in view of (29) and (30), imply that $\phi_k,\xi_k\in\mathscr{L}_{\infty e}$ for all $k\in\mathbb{Z}_+$, and hence $\bar{u}_k,\hat{e}_{1,k},e_{1,k}\in\mathscr{L}_{\infty e}$ for all $k\in\mathbb{Z}_+$. Hence, $y_k\in\mathscr{L}_{\infty e}$ for all $k\in\mathbb{Z}_+$ since $y_d(t)$ is bounded. Considering system (3) under the control law (26), and keeping in mind that $y_k,\theta_k\in\mathscr{L}_{\infty e}$ for all $k\in\mathbb{Z}_+$, it is clear that $\omega_{1,k},\omega_{2,k}\in\mathscr{L}_{\infty e}$ for all $k\in\mathbb{Z}_+$ since there is no finite escape-time for the solutions of the linear time-varying system (3) and (26). Since $r_f(t)$ is bounded, it is clear that $\Omega_k\in\mathscr{L}_{\infty e}$, and hence $u_k\in\mathscr{L}_{\infty e}$. Consequently, one can conclude that all signals are bounded $\forall k\in\mathbb{Z}_+$, $\forall t\in[0,T]$.

One can also conclude from (35) that

$$\lim_{k\to\infty}\bar{\rho}_k(t)=\lim_{k\to\infty}\bar{\theta}_k(t)=\lim_{k\to\infty}\varepsilon_k(t)=0 \tag{36}$$

for all $t\in[0,T]$. Since the previous development is valid for $\sup_{t\in[0,T]}|\varepsilon_{k-1}(t)|>\sigma$, (36) is not true. In fact, one can only conclude that $\lim_{k\to\infty}|\varepsilon_k(t)|\leqslant\sigma$, $\forall t\in[0,T]$. Once $\sup_{t\in[0,T]}|\varepsilon_k(t)|\leqslant\sigma$, according to (29), $\theta_k(t)$ becomes constant, and hence from (31), $\bar{u}_k=\theta_k^{\mathrm{T}}\phi_k$ and hence $\xi_k=0$ which implies that $\lim_{k\to\infty}|e_{1,k}(t)|\leqslant\kappa\sigma$, $\forall t\in[0,T]$. □

[§]For $k=0$ the control scheme (26)–(28) is nothing else but the classical MRAC, and hence the proof of boundedness of all signals is shown in References [1, 2].

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Remark 5
Note that, for $k = 0$, the control scheme proposed in Theorem 2 reduces to the standard MRAC schemes proposed in References [1, 2]. In fact, $m_0^2 = 1 + \bar{u}_0^2 + \phi_0^{\mathrm{T}}\phi_0$ has been used in Reference [1] and $m_0^2 = 1 + \bar{\phi}_0^{\mathrm{T}}\bar{\phi}_0$ has been used in Reference [2].

Remark 6
It is worth noting that the saturation used for θ_k is required for a technical reason in the proof. It allows to ensure that $\theta_k(t)$ becomes constant when the augmented tracking error ε_k is sufficiently small. In this case, the augmented tracking error becomes the real tracking error $e_{1,k}$ since $\xi_k(t) = 0$. On the other hand, the saturation used for ρ_k is not necessary. In fact, we stop the learning for ρ_k because it has no effect on the system behaviour once θ_k is constant.

Remark 7
Although not explicitly mentioned in our theorems, the proposed MRAILC schemes are able to handle iteration-varying desired trajectories. In other words, the learning process remains effective even if the desired trajectory (or the reference model) is changing from iteration to iteration. This is one important advantage of this Lyapunov-based framework with respect to the traditional contraction mapping-based frameworks (see, for instance, Reference [13]).

6. SIMULATION RESULTS

In this section, we consider three examples.

Example 1

$$G_p(s) = \frac{s+1}{s^2 - 10s + 1}, \quad G_m(s) = \frac{1}{s+1}$$

with $r_f(t)$ being a unit step input. The auxiliary variables $w_{1,k}$ and $w_{2,k}$ are given by

$$w_{1,k} = \frac{1}{s+10}[u_k], \quad w_{2,k} = \frac{1}{s+10}[y_k]$$

The matrix Γ is chosen as a $\Gamma = 10I_{4\times 4}$. The time interval is taken as $[0, 8s]$ and the initial conditions for the adaptive law at the first iteration are chosen to be zero.

Figure 1 shows the evolution of the Sup-norm of the tracking error with respect to the iteration number. Figure 2 shows the performance of the standard MRAC. Figure 3 shows the transient performance improvement over the iterations.

Example 2

$$G_p(s) = \frac{2s+5}{s^3 + 6s^2 + 7s - 4}, \quad G_m(s) = \frac{2s+5}{s^3 + 6s^2 + 11s + 6}$$

with $r_f(t)$ being a unit step input. The auxiliary variables $w_{1,k}$ and $w_{2,k}$ are given by

$$w_{1,k} = \left(\frac{s}{2s^2 + 15s + 25}, \frac{1}{2s^2 + 15s + 25}\right)^{\mathrm{T}}[u_k], \quad w_{2,k} = \left(\frac{s}{2s^2 + 15s + 25}, \frac{1}{2s^2 + 15s + 25}\right)^{\mathrm{T}}[y_k]$$

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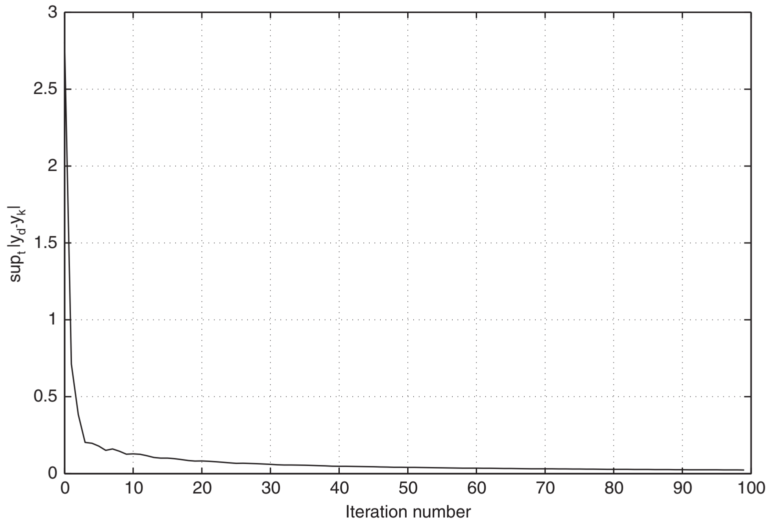

Figure 1. Example 1: $\sup_{t\in[0,8]}|y_d(t)-y_k(t)|$ with respect to the iteration number k.

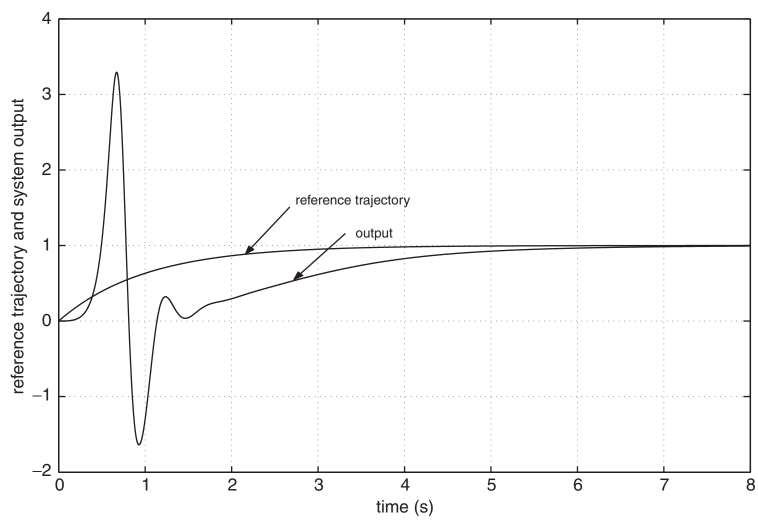

Figure 2. Example 1: Reference trajectory and system output with the MRAC (i.e. $k=0$).

$\kappa=0.1$, $\Gamma=10I_{6\times 6}$, $\gamma=0.1$ and $\sigma=0.05$. The time interval is taken as $[0,8s]$ and the initial conditions for the adaptive law at the first iteration are chosen to be zero. At the first iteration, i.e. for $k=0$, we use $m_0^2=1+\bar{u}_0^2+\phi_0^{\mathrm{T}}\phi_0$.

Figure 4 shows the evolution of the Sup-norm of the tracking error with respect to the iteration number using Theorem 2.

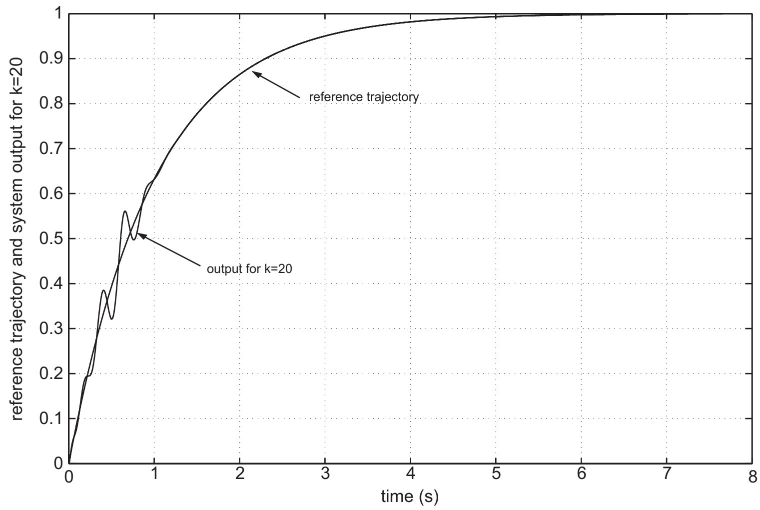

Figure 3. Example 1: Reference trajectory and system output at the 20th iteration.

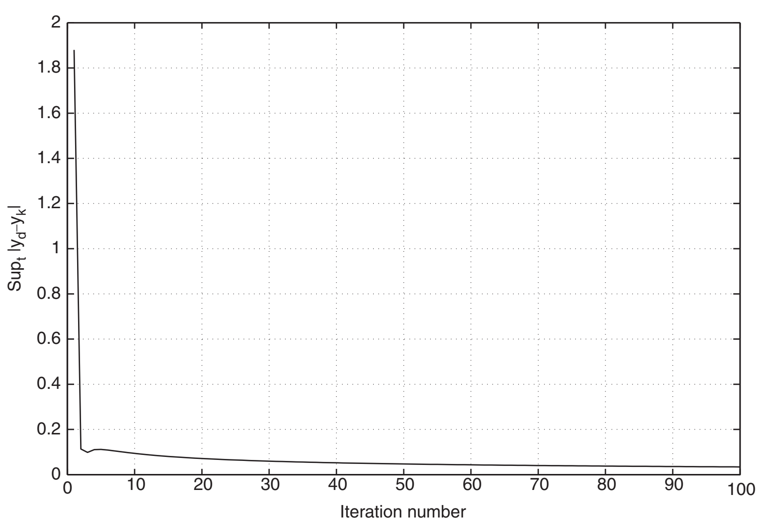

Figure 4. Example 2: $\sup_{t\in[0,8]} |y_d(t) - y_k(t)|$ with respect to the iteration number k.

Example 3

In this example, we show the effectiveness of our algorithms when the desired trajectory is iteration-varying. To this end, we consider the system of Example 1 with the following iteration-varying reference model:

$$G_{m,k}(s) = \frac{k+1}{s+1}$$

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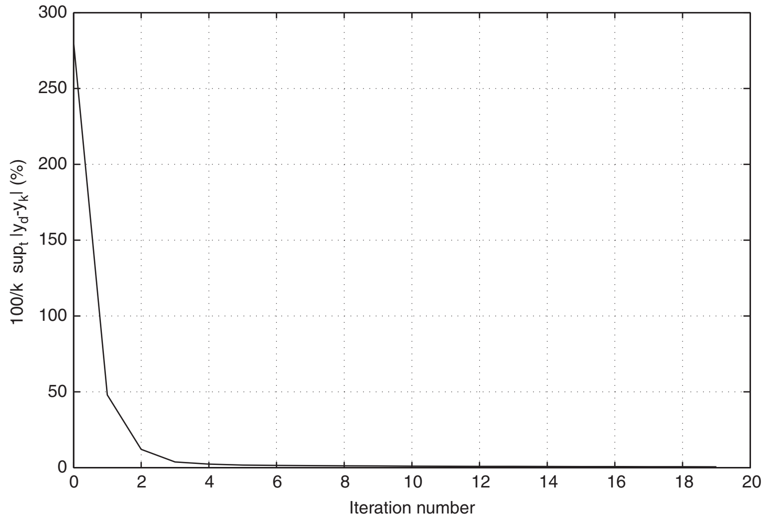

Figure 5. Example 3: $(100/k)\sup_{t\in[0,8]}|y_d(t) - y_k(t)|$ (%) with respect to the iteration number k.

with $k \in \{0, 1, \ldots, 19\}$. The auxiliary variables $w_{1,k}$, $w_{2,k}$ and the control gains are taken as in Example 1. The time interval is taken as $[0, 8s]$ and the initial conditions for the adaptive law at the first iteration are chosen to be zero. Figure 5 shows the evolution of the Sup-norm percentage of the tracking error with respect to the iteration number.

7. CONCLUSION

In this paper, we proposed a straightforward extension of standard MRAC schemes to linear repetitive systems in order to improve the transient tracking performance through iterative learning. In fact, this was made possible through the introduction of a parametric adaptation law along the iteration-axis, obtained directly from the continuous-time parametric adaptation law used in standard MRAC schemes. In the proposed approach, at the first iteration, i.e. for $k = 0$, we apply a standard MRAC scheme. Thereafter, i.e. for $k \geqslant 1$, the parameter estimates are iteratively updated in order to refine the output response and enhance the tracking performance from iteration to iteration. The proposed MRAILC scheme achieves a pointwise convergence of the tracking error to zero (in the case of systems with relative degree one), or to a prescribed small domain around zero (for systems with higher relative degree), over the whole finite time interval, when the number of iterations tends to infinity. In contrast to existing contraction mapping-based ILC schemes, the proposed control strategy does not require the use of the output time derivatives and can handle systems with multiple tracking objectives (i.e. the desired trajectory can be modified from iteration to iteration).

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