



On consensus algorithms for double-integrator dynamics without velocity measurements and with input constraints[☆]

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ABSTRACT

This note deals with consensus strategy design for double-integrator dynamics. Specifically, we consider the case where the control inputs are required to be *a priori* bounded and the velocity (second state) is not available for feedback. Two different design methods are proposed. First, based on the auxiliary system approach, we propose a consensus algorithm that extends some of the existing results in the literature to account for actuator saturations and the lack of velocity measurement. The proposed velocity-free control scheme, using local information exchange, achieves consensus among the team members with an *a priori* bounded control law, whose upper bound depends on the number of neighbors of the vehicle. Second, we propose another approach based on the use of a high order dynamic auxiliary system such that the upper bound of the control law is independent of the number of neighbors of the vehicle, and the performance of the closed loop system is improved in terms of the response damping. Finally, simulation results are provided to illustrate the effectiveness of the proposed algorithms.

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1. Introduction

Consensus algorithms have received great interest in the control community, leading to several results that have been applied to a variety of problems related to the cooperative control of multi-vehicle systems such as flocking, rendezvous and formation control (see [1,2] and references therein). A team of agents is said to achieve consensus if all members of the team reach an agreement on a common final value using local information exchange. Consensus algorithms for single-integrator kinematics have been widely studied in the literature, [1–6] to name a few, and several interesting results have been obtained using properties from graph theory. These results have been extended to double-integrator dynamics in [7–10] where it was shown that consensus algorithms for double-integrator dynamics are more challenging than first order kinematics. Several variants of the proposed algorithms have been applied to flocking [11,12], rigid body attitude synchronization [13,14] and formation control [15,16].

While most consensus algorithms for double-integrator dynamics rely on the availability of the full state for feedback, only a

few works have been done when velocity information is not available. In fact, this problem is faced when vehicles are not equipped with velocity sensors, to save cost, space and weight, or velocity is not precisely measured. Another important problem that often arises in practical applications, is to design algorithms that account for actuator saturations. The problem becomes quite serious when the number of vehicles in the team is large and the information flow is high, *i.e.*, each vehicle has a large number of neighbors. This leads to a high control effort for each vehicle that causes actuator saturations. In [15], formation control strategies for multirobot formation maneuvers are discussed. The authors present two control schemes that respectively account for actuator saturation and consider the lack of relative velocity measurements. In this work, the communication flow between vehicles is restricted to a bidirectional ring. The author in [8] extends the results in [15] to a more general undirected communication topology, and presents some consensus algorithms for double-integrator dynamics without relative velocity measurements. However, in both works, the velocity-free consensus algorithms do not take into consideration actuator saturations.

The main contribution of this note is to propose consensus algorithms for double-integrator dynamics without velocity measurements and in the presence of input saturation constraints. To the best of our knowledge, this work is the first dealing with these two issues simultaneously. We discuss two conceptually different design methodologies that achieve consensus, with and without reference velocity, under a fixed and bidirectional communication topology. The first approach discussed in this note

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extends the velocity-free consensus algorithm proposed in [8] to account for actuator saturations. This approach is based on the introduction of an auxiliary system for each agent, acting as a reduced order observer. The input of this auxiliary system is determined using the relative positions of the vehicles and its output is used in the control law to generate the necessary damping in the absence of velocity signals. As a result, consensus is achieved with an *a priori* bounded control law using only position relative errors. Consequently, the communication requirements between the team members is considerably reduced as compared to the full information case.

As will be clear throughout the paper, the first approach suffers from some limitations. First, the upper bound of the control law depends on the number of neighbors of each member of the team. This is generally not desirable especially when each vehicle has a large number of neighbors, since a trade-off between maintaining the control upper bound and good performance can hardly be achieved. Second, as most auxiliary systems-based methods, this approach suffers from the lack of sufficient damping, and the system response generally presents transient oscillations before reaching consensus. To solve these problems, we propose a second consensus algorithm based on a high order auxiliary system. The proposed scheme guarantees that the control input of each vehicle can be *a priori* bounded regardless of the number of its neighbors. Furthermore, additional damping is added such that transient oscillations can be reduced and/or eliminated. In this approach, the control objective is achieved by forcing the vehicles' relative positions to track the relative errors of the auxiliary variables. Thereafter, the auxiliary variables are forced to converge to zero asymptotically, guiding the relative positions between all vehicles in the team to zero.

2. Problem formulation and notations

Consider n -“vehicles” with double-integrator dynamics given by

$$\dot{\mathbf{r}}_i = \mathbf{v}_i, \quad \dot{\mathbf{v}}_i = \mathbf{u}_i, \quad \text{for } i \in \mathcal{N} \quad (1)$$

where $\mathbf{r}_i \in \mathbb{R}^m$ and $\mathbf{v}_i \in \mathbb{R}^m$ are respectively the position and velocity of the i th vehicle, and \mathbf{u}_i is the control input, and $\mathcal{N} \triangleq \{1, \dots, n\}$. To design consensus algorithms, vehicles must communicate some of their states with each other. In this work, we assume that the information flow between members of the team is fixed and undirected. Then, it is natural to describe the information flow between vehicles using weighted graphs. A weighted undirected graph, \mathcal{G} , consists of the triplet $(\mathcal{N}, \mathcal{E}, \mathcal{K})$, with \mathcal{N} being the set of nodes or vertices, describing the set of vehicles in the team, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ the set of unordered pairs of nodes, called edges, and $\mathcal{K} = [k_{ij}] \in \mathbb{R}^{n \times n}$ is a weighted adjacency matrix. An edge (i, j) indicates that vehicles i and j are neighbors and can obtain information from one another. The weighted adjacency matrix of a weighted undirected graph is defined such that $k_{ij} = k_{ji} > 0$ for $(i, j) \in \mathcal{E}$, and $k_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. If there is a path between any two distinct nodes of a weighted undirected graph \mathcal{G} , then \mathcal{G} is said to be connected. For more details on graph properties, the reader is referred to [17].

The objective of our work is to design control algorithms for a group of vehicles such that consensus is reached on their final states and all vehicles track a desired velocity, i.e., $\mathbf{r}_i(t) \rightarrow \mathbf{r}_j(t)$ and $\mathbf{v}_i(t) \rightarrow \mathbf{v}_d(t)$ for all $i, j \in \mathcal{N}$, with $\mathbf{v}_d(t)$ being a reference velocity. Also, we aim to extend our results to solve the consensus problem without reference velocity, where it is required that $\mathbf{r}_i(t) \rightarrow \mathbf{r}_j(t)$ and $\mathbf{v}_i(t) \rightarrow \mathbf{v}_j(t)$ for all $i, j \in \mathcal{N}$. In this work, we assume that the absolute and relative velocities are not measurable, and all vehicles are subject to input saturation constraints such that $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$ for $i \in \mathcal{N}$.

3. Control design I

In this section, we propose a consensus algorithm without relative velocity measurements for (1) subject to input saturation constraints. Our result is stated in the following theorem.

Theorem 1. Consider a group of n -vehicles modeled as in (1), with the following control input

$$\mathbf{u}_i = \dot{\mathbf{v}}_d(t) - k_i^v \tanh(\lambda^\psi (\mathbf{r}_i - \boldsymbol{\psi}_i)) - \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) + \dot{\boldsymbol{\phi}}_i, \quad \text{for } i \in \mathcal{N}, \quad (2)$$

where λ^r and λ^ψ are strictly positive scalar gains, $k_{ij} \geq 0$ is the (i, j) th entry of the weighted adjacency matrix \mathcal{K} of the communication graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{K})$, characterizing the information flow between vehicles, and the function $\tanh(\cdot)$ is defined element-wise for a vector. The scalar gain k_i^v is defined such that $k_i^v > 0$ if $i \in \mathcal{I}$, and $k_i^v = 0$ otherwise, where the set $\mathcal{I} \neq \emptyset$ is a subset of \mathcal{N} . The vectors $\boldsymbol{\phi}_i \in \mathbb{R}^m$ and $\boldsymbol{\psi}_i \in \mathbb{R}^m$ are respectively given by

$$\dot{\boldsymbol{\phi}}_i = -k_i^\phi \tanh(\lambda^\phi \boldsymbol{\phi}_i) - k_i^\psi \tanh(\lambda^\psi (\mathbf{r}_i - \boldsymbol{\psi}_i)) - \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)), \quad \text{for } i \in \mathcal{N}, \quad (3)$$

$$\dot{\boldsymbol{\psi}}_i = \mathbf{v}_d + k_i^\psi (\mathbf{r}_i - \boldsymbol{\psi}_i), \quad \text{for } i \in \mathcal{I}, \quad (4)$$

where λ^ϕ , k_i^ϕ and k_i^ψ are strictly positive scalar gains and $\boldsymbol{\phi}_i(0)$ and $\boldsymbol{\psi}_i(0)$ can be selected arbitrarily. Assume that the desired velocity and its first time derivative are bounded. Let the controller gains satisfy

$$2 \left(k_i^v + \sum_{j=1}^n k_{ij} \right) + k_i^\phi \leq \mathbf{u}_{\max} - \|\dot{\mathbf{v}}_d\|_\infty, \quad \text{for } i \in \mathcal{N}, \quad (5)$$

for any $\mathbf{u}_{\max} > 0$, and let the communication graph \mathcal{G} be connected.¹ Then

- (i) $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$ for all $i \in \mathcal{N}$,
- (ii) the signals \mathbf{v}_i , $\boldsymbol{\phi}_i$, $(\mathbf{r}_i - \mathbf{r}_j)$, for all $i, j \in \mathcal{N}$, and $(\mathbf{r}_i - \boldsymbol{\psi}_i)$, for $i \in \mathcal{I}$, are globally bounded,
- (iii) $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = \lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_d(t)) = 0$ for all $i, j \in \mathcal{N}$.

Proof. First, one can easily check that the control input (2) with (3)–(4) is bounded as $\|\mathbf{u}_i\|_\infty \leq 2(k_i^v + \sum_{j=1}^n k_{ij}) + k_i^\phi + \|\dot{\mathbf{v}}_d\|_\infty$. Hence (i) follows from condition (5).

Let the velocity tracking error for each vehicle be defined as: $\tilde{\mathbf{v}}_i = (\mathbf{v}_i - \mathbf{v}_d)$, for $i \in \mathcal{N}$, and consider the Lyapunov function candidate

$$V_1 = \frac{1}{2} \sum_{i=1}^n (\tilde{\mathbf{v}}_i - \boldsymbol{\phi}_i)^T (\tilde{\mathbf{v}}_i - \boldsymbol{\phi}_i) + \frac{1}{2} \sum_{i=1}^n \boldsymbol{\phi}_i^T \boldsymbol{\phi}_i + \sum_{i=1}^n \frac{k_i^v}{\lambda^\psi} \mathbf{1}_m^T \log(\cosh(\lambda^\psi (\mathbf{r}_i - \boldsymbol{\psi}_i))) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{k_{ij}}{\lambda^r} \mathbf{1}_m^T \log(\cosh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j))), \quad (6)$$

where $\mathbf{1}_m \in \mathbb{R}^m$ is the vector with all elements equal to one and the functions $\log(\cdot)$ and $\cosh(\cdot)$ are defined element-wise for a vector. The time derivative of V_1 along the closed loop dynamics (1) with (2) is given by

¹ The communication graph \mathcal{G} is said to be connected if every vehicle can communicate with at least one other vehicle in the team.

$$\begin{aligned}
\dot{V}_1 = & \sum_{i=1}^n (\tilde{\mathbf{v}}_i - \phi_i)^T \left(-k_i^v \tanh(\lambda^\psi (\mathbf{r}_i - \psi_i)) \right. \\
& \left. - \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) \right) + \sum_{i=1}^n \phi_i^T \dot{\phi}_i \\
& + \sum_{i=1}^n k_i^v (\tilde{\mathbf{v}}_i + \mathbf{v}_d - \dot{\psi}_i)^T \tanh(\lambda^\psi (\mathbf{r}_i - \psi_i)) \\
& + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n k_{ij} (\mathbf{v}_i - \mathbf{v}_j)^T \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)). \quad (7)
\end{aligned}$$

Motivated by the result of Lemma 3.1 in [8], and using the relation $(\tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_j) = (\mathbf{v}_i - \mathbf{v}_j)$, with the fact that the information flow between vehicles is undirected, i.e., $k_{ij} = k_{ji}$, we can easily show that

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n k_{ij} (\mathbf{v}_i - \mathbf{v}_j)^T \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) \\
& = \sum_{i=1}^n \sum_{j=1}^n k_{ij} \tilde{\mathbf{v}}_i^T \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)). \quad (8)
\end{aligned}$$

Then, we can write

$$\begin{aligned}
\dot{V}_1 = & \sum_{i=1}^n \phi_i^T \left(\dot{\phi}_i + k_i^v \tanh(\lambda^\psi (\mathbf{r}_i - \psi_i)) \right. \\
& \left. + \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) \right) \\
& + \sum_{i=1}^n k_i^v (\mathbf{v}_d - \dot{\psi}_i)^T \tanh(\lambda^\psi (\mathbf{r}_i - \psi_i)). \quad (9)
\end{aligned}$$

Using (3)–(4) we obtain

$$\begin{aligned}
\dot{V}_1 = & - \sum_{i=1}^n k_i^\phi \phi_i^T \tanh(\lambda^\phi \phi_i) \\
& - \sum_{i=1}^n k_i^v k_i^\psi (\mathbf{r}_i - \psi_i)^T \tanh(\lambda^\psi (\mathbf{r}_i - \psi_i)), \quad (10)
\end{aligned}$$

which is negative semi-definite, and we can conclude that \mathbf{v}_i , ϕ_i , for $i \in \mathcal{N}$, $(\mathbf{r}_i - \mathbf{r}_j)$, for $(i, j) \in \mathcal{E}$ and $(\mathbf{r}_i - \psi_i)$, for $i \in \mathcal{L}$, are globally bounded. Hence, point (ii) follows from the assumption that the undirected communication graph is connected.

Using the above boundedness results, we can see from (4) that $\dot{\psi}_i$ is bounded, for $i \in \mathcal{L}$. Since $\dot{\phi}_i$ is bounded, for $i \in \mathcal{N}$, we conclude that \dot{V}_1 is bounded, and invoking Barbālat Lemma we can conclude that $\lim_{t \rightarrow \infty} \phi_i(t) = 0$ for $i \in \mathcal{N}$ and $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \psi_i(t)) = 0$ for $i \in \mathcal{L}$. In addition, since $\dot{\phi}_i$, \mathbf{v}_i , for $i \in \mathcal{N}$ and $\dot{\psi}_i$, for $i \in \mathcal{L}$, are bounded, we know that $\dot{\phi}_i$ is bounded, and since we have already shown that $\lim_{t \rightarrow \infty} \phi_i(t) = 0$, we know by Barbālat Lemma that $\lim_{t \rightarrow \infty} \dot{\phi}_i(t) = 0$, for $i \in \mathcal{N}$. Therefore, we conclude from (3) that

$$\lim_{t \rightarrow \infty} \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) = 0, \quad \text{for } i \in \mathcal{N}. \quad (11)$$

Multiplying the above set of equations by \mathbf{r}_i and taking the sum over i , (11) will be equivalent to

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n k_{ij} \mathbf{r}_i^T \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) \\
& = \frac{1}{2} \lim_{t \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n k_{ij} (\mathbf{r}_i - \mathbf{r}_j)^T \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) = 0, \quad (12)
\end{aligned}$$

where we have used the fact that the information flow between vehicles is undirected, i.e., $k_{ij} = k_{ji}$. As a result, we can conclude that $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = 0$ for all $(i, j) \in \mathcal{E}$. Since the undirected communication graph is assumed connected, this result is valid for all $i, j \in \mathcal{N}$. Furthermore, we can see from (1) and (2) that $\dot{\mathbf{v}}_i$ is bounded for $i \in \mathcal{N}$, and hence we can conclude by Barbālat Lemma that $\lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_j(t)) = 0$ for all $i, j \in \mathcal{N}$.

Exploiting the above results, we can see that $\dot{\psi}_i = \dot{\mathbf{v}}_d + k_i^\psi (\mathbf{v}_i - \dot{\psi}_i)$ is bounded for $i \in \mathcal{L}$ and \mathbf{u}_i in (2) is bounded for $i \in \mathcal{N}$. Therefore, we know that $(\dot{\mathbf{r}}_i - \dot{\psi}_i)$ is bounded for $i \in \mathcal{L}$. Then, from Barbālat Lemma, and since $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \psi_i(t)) = 0$ for $i \in \mathcal{L}$, we conclude that $\lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \dot{\psi}_i(t)) = 0$ for $i \in \mathcal{L}$. Consequently, we know from (4) that $\lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_d(t)) = 0$ for $i \in \mathcal{L}$. Finally, since the set $\mathcal{L} \neq \emptyset$, i.e., contains at least one element, and $\lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_j(t)) = 0$, for all $i, j \in \mathcal{N}$, we can conclude that $\lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_d(t)) = 0$ for all $i \in \mathcal{N}$, and this ends the proof. \square

It is worth noting that the auxiliary system, with output ϕ_i , is used in the above control scheme to drive the relative positions and the relative velocities to zero without velocity measurements. The second auxiliary system, with output ψ_i , is used to drive the velocity of at least one vehicle in the team to the desired velocity. As a result, all vehicles reach consensus on their final states and track the desired final velocity. Note that the second auxiliary system is implemented for only some vehicles in the team, as defined by the set \mathcal{L} . Therefore, the variable ψ_i is not defined for $i \notin \mathcal{L}$, which does not change our results since $k_i^v = 0$ in this case. Also, note that the desired velocity needs to be available to only some vehicles in the team. However, if the desired velocity is time varying, all vehicles must have access to its time derivative.

In the case where no reference velocity is assigned to the team, and it is desirable that consensus is achieved, i.e., $\mathbf{r}_i(t) \rightarrow \mathbf{r}_j(t)$ and $\mathbf{v}_i(t) \rightarrow \mathbf{v}_j(t)$ asymptotically for all $i, j \in \mathcal{N}$, we propose the control strategy stated in the following corollary.

Corollary 1. Consider a group of n -vehicles modeled as in (1), with the following control input

$$\begin{cases} \mathbf{u}_i = - \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)) + \dot{\phi}_i, \\ \dot{\phi}_i = -k_i^\phi \tanh(\lambda^\phi \phi_i) - \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)), \end{cases} \quad (13)$$

for $i \in \mathcal{N}$, where λ^r , λ^ϕ , k_i^ϕ and k_{ij} are defined as in Theorem 1, and $\phi_i(0)$ can be selected arbitrarily. Let the controller gains satisfy

$$2 \sum_{j=1}^n k_{ij} + k_i^\phi \leq \mathbf{u}_{\max}, \quad \text{for } i \in \mathcal{N}, \quad (14)$$

for any $\mathbf{u}_{\max} > 0$, and let the communication graph \mathcal{G} be connected. Then

- (i) $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$ for $i \in \mathcal{N}$,
- (ii) the signals \mathbf{v}_i , ϕ_i and $(\mathbf{r}_i - \mathbf{r}_j)$ are globally bounded for all $i, j \in \mathcal{N}$,
- (iii) $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = \lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_j(t)) = 0$ for all $i, j \in \mathcal{N}$.

Proof. First, note that the control input (13) is bounded as $\|\mathbf{u}_i\|_\infty \leq 2 \sum_{j=1}^n k_{ij} + k_i^\phi$, which, under the constraint on the control gains (14), yields (i). Using the following Lyapunov function candidate

$$\begin{aligned}
V_2 = & \frac{1}{2} \sum_{i=1}^n (\mathbf{v}_i - \phi_i)^T (\mathbf{v}_i - \phi_i) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{k_{ij}}{\lambda^r} \mathbf{1}_m^T \\
& \times \log(\cosh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j))) + \frac{1}{2} \sum_{i=1}^n \phi_i^T \phi_i, \quad (15)
\end{aligned}$$

and following similar arguments as in the proof of [Theorem 1](#), with the help of Barbălat Lemma, the rest of the corollary can be proven. \square

Remark 1. Note that the control scheme (13) guarantees that all vehicles achieve consensus with constant final velocity since \mathbf{v}_i is globally bounded and $\lim_{t \rightarrow \infty} \mathbf{u}_i(t) = 0$ for $i \in \mathcal{N}$.

Remark 2. Notice from the above consensus algorithms that only a single auxiliary system, with output ϕ_i , is implemented for each vehicle in the team to achieve the result of [Corollary 1](#). Whereas, to guarantee the results of [Theorem 1](#), we need to implement an additional auxiliary system, with output ψ_i , for at least one vehicle in the team. In both cases, only the position vector \mathbf{r}_i is transmitted between vehicles in the team.

It is important to mention that the control scheme in [Theorem 1](#) can be modified to guarantee the same results using only a single auxiliary system for each vehicle. This can be seen from the control law

$$\mathbf{u}_i = \dot{\mathbf{v}}_d(t) - k_i^v \tanh(\lambda^\psi (\mathbf{r}_i - \boldsymbol{\psi}_i)) - \sum_{j=1}^n k_{ij} \tanh(\lambda^r (\mathbf{r}_i - \mathbf{r}_j)), \quad \text{for } i \in \mathcal{N}, \quad (16)$$

where $\boldsymbol{\psi}_i$ is given in (4), with $\mathcal{I} = \mathcal{N}$, and the control gains are defined as in [Theorem 1](#). In fact, following similar steps as in the proof of [Theorem 1](#), we can easily show that (16) achieves consensus with reference velocity. However, the auxiliary system with output ϕ_i is necessary to achieve consensus without reference velocity, using only the relative positions of the vehicles in the team.

Remark 3. The proposed control schemes in (2)–(4), with $\mathbf{v}_d = 0$, and (13) extend respectively the results in [Corollary 4.2](#) and [Theorem 4.1](#) in [8] to account for actuator saturations. The main difference between the two works is the adopted design method, namely the choice of the Lyapunov function. In fact, the Lyapunov function used in [8] does not facilitate the design of an *a priori* bounded control law for our system. Furthermore, and as was shown in the result of [Theorem 1](#), the proposed control scheme can handle time-varying trajectories, which is not obvious using the design method proposed in [8].

Remark 4. Although the presented control schemes in this section achieve our control objectives, they suffer from two limitations that are shared with the design of [8]. First, we can see from (2)–(4) and (13) that the upper bounds of the proposed control schemes depend on the number of neighbors of each vehicle. This constitutes a constraint in tuning the above controllers. In fact, it is generally hard to obtain a trade-off between guaranteeing the controller upper bounds with achieving an acceptable/good transient performance. This problem becomes more important when the number of neighbors of each vehicle is large and \mathbf{u}_{\max} is small. Second, since the outputs of the auxiliary systems are used in the control law instead of the missing velocity vectors, the proposed control schemes in this section suffer from the oscillatory transient response of the states (lack of sufficient damping). The above mentioned limitations will be obviated in a new consensus design that will be the subject of the next section.

4. Control design II—high order auxiliary systems

In this section, we extend the results in the previous section, and present a consensus algorithm for the system that solves the problems discussed above. To this end, we introduce the following new variables for each vehicle

$$\xi_i := \mathbf{r}_i - \boldsymbol{\theta}_i, \quad \mathbf{z}_i := \dot{\xi}_i = \mathbf{v}_i - \dot{\boldsymbol{\theta}}_i. \quad (17)$$

The primary role of the variable $\boldsymbol{\theta}_i \in \mathbb{R}^m$ is to enable the design of an *a priori* bounded control input for each vehicle, where the upper bound is independent of the number of neighbors. We propose the following control law

$$\begin{cases} \mathbf{u}_i = \dot{\mathbf{v}}_d - \Phi_i, \\ \ddot{\boldsymbol{\theta}}_i = -\Phi_i - \dot{\phi}_i + k_i^v (\xi_i - \boldsymbol{\psi}_i) + \sum_{j=1}^n k_{ij} (\xi_i - \xi_j), \end{cases} \quad (18)$$

with

$$\Phi_i = k_{\theta_{1i}} \tanh(\boldsymbol{\theta}_i) + k_{\theta_{2i}} \tanh(\dot{\boldsymbol{\theta}}_i), \quad (19)$$

for $i \in \mathcal{N}$, where $\phi_i \in \mathbb{R}^m$ and $\boldsymbol{\psi}_i \in \mathbb{R}^m$ are design variables to be determined later, $k_{\theta_{1i}}$ and $k_{\theta_{2i}}$ are strictly positive scalar gains, k_{ij} are defined as in [Theorem 1](#), and $\boldsymbol{\theta}_i(0)$ and $\dot{\boldsymbol{\theta}}_i(0)$ can be selected arbitrarily. The scalar gain k_i^v is defined such that $k_i^v > 0$ if $i \in \mathcal{I}$, and $k_i^v = 0$ otherwise, where the set $\mathcal{I} \neq \emptyset$ is a subset of \mathcal{N} . We assume that the desired velocity and its first time derivative are bounded.

It is important to notice that the control input is bounded as

$$\|\mathbf{u}_i\|_\infty \leq k_{\theta_{1i}} + k_{\theta_{2i}} + \|\dot{\mathbf{v}}_d\|_\infty, \quad \text{for } i \in \mathcal{N}, \quad (20)$$

regardless of the number of neighbors of the i th vehicle. Then, the proposed control law (18) accounts for actuator saturations if the controller gains satisfy

$$k_{\theta_{1i}} + k_{\theta_{2i}} \leq \mathbf{u}_{\max} - \|\dot{\mathbf{v}}_d\|_\infty, \quad \text{for } i \in \mathcal{N}, \quad (21)$$

for any $\mathbf{u}_{\max} > 0$. Before stating our result, we need the following lemma:

Lemma 1. Consider the second order system

$$\ddot{\boldsymbol{\theta}}_i = -k_{\theta_{1i}} \tanh(\boldsymbol{\theta}_i) - k_{\theta_{2i}} \tanh(\dot{\boldsymbol{\theta}}_i) + \boldsymbol{\eta}_i(t). \quad (22)$$

with $\boldsymbol{\theta}_i \in \mathbb{R}^m$, $k_{\theta_{1i}}$ and $k_{\theta_{2i}}$ are positive scalar gains and the function $\tanh(\cdot)$ is defined element-wise for a vector. If $\boldsymbol{\eta}_i(t)$ is bounded for all time and $\lim_{t \rightarrow \infty} \boldsymbol{\eta}_i(t) = 0$, then $\boldsymbol{\theta}_i$ and $\dot{\boldsymbol{\theta}}_i$ are bounded and $\lim_{t \rightarrow \infty} \boldsymbol{\theta}_i(t) = \lim_{t \rightarrow \infty} \dot{\boldsymbol{\theta}}_i(t) = 0$.

Proof. See [Appendix](#). \square

Now, we can state our main result in the following theorem.

Theorem 2. Consider a group of n -vehicles modeled as in (1), with the control input (18). Let

$$\dot{\phi}_i = -k_i^\phi \phi_i - k_i^v (\xi_i - \boldsymbol{\psi}_i) - \sum_{j=1}^n k_{ij} (\xi_i - \xi_j), \quad \text{for } i \in \mathcal{N}, \quad (23)$$

$$\dot{\boldsymbol{\psi}}_i = \mathbf{v}_d + k_i^\psi (\xi_i - \boldsymbol{\psi}_i), \quad \text{for } i \in \mathcal{I}, \quad (24)$$

with k_i^ϕ and k_i^ψ defined as in [Theorem 1](#), and $\phi_i(0)$ and $\boldsymbol{\psi}_i(0)$ can be selected arbitrarily. Let the controller gains satisfy condition (21) and let the communication graph \mathcal{G} be connected. Then

- (i) $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$ for all $i \in \mathcal{N}$,
- (ii) the signals \mathbf{v}_i , ϕ_i , $\boldsymbol{\theta}_i$, $\dot{\boldsymbol{\theta}}_i$, $(\mathbf{r}_i - \mathbf{r}_j)$, for all $i, j \in \mathcal{N}$, and $(\mathbf{r}_i - \boldsymbol{\psi}_i)$, for $i \in \mathcal{I}$, are globally bounded,
- (iii) $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = \lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_d(t)) = 0$ for all $i, j \in \mathcal{N}$.

Proof. The result (i) is obvious from (20) and condition (21). Now, let us prove (ii) and (iii). Let $\tilde{\mathbf{z}}_i = (\mathbf{z}_i - \mathbf{v}_d)$, for $i \in \mathcal{N}$, and consider the Lyapunov function candidate

$$\begin{aligned}
V_3 &= \frac{1}{2} \sum_{i=1}^n (\tilde{\mathbf{z}}_i - \phi_i)^T (\tilde{\mathbf{z}}_i - \phi_i) \\
&+ \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n k_{ij} (\xi_i - \xi_j)^T (\xi_i - \xi_j) \\
&+ \frac{1}{2} \sum_{i=1}^n \phi_i^T \phi_i + \frac{1}{2} \sum_{i=1}^n k_i^v (\xi_i - \psi_i)^T (\xi_i - \psi_i), \quad (25)
\end{aligned}$$

whose time derivative along the system dynamics (1) with (17) gives

$$\begin{aligned}
\dot{V}_3 &= \sum_{i=1}^n (\tilde{\mathbf{z}}_i - \phi_i)^T (\mathbf{u}_i - \dot{\mathbf{v}}_d - \ddot{\theta}_i - \dot{\phi}_i) \\
&+ \sum_{i=1}^n \sum_{j=1}^n k_{ij} \tilde{\mathbf{z}}_i^T (\xi_i - \xi_j) + \sum_{i=1}^n \phi_i^T \dot{\phi}_i \\
&+ \sum_{i=1}^n k_i^v (\tilde{\mathbf{z}}_i + \mathbf{v}_d - \dot{\psi}_i)^T (\xi_i - \psi_i), \quad (26)
\end{aligned}$$

where we have used the relation $\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n k_{ij} (\tilde{\mathbf{z}}_i - \tilde{\mathbf{z}}_j)^T (\xi_i - \xi_j) = \sum_{i=1}^n \sum_{j=1}^n k_{ij} \tilde{\mathbf{z}}_i^T (\xi_i - \xi_j)$, with $(\tilde{\mathbf{z}}_i - \tilde{\mathbf{z}}_j) = (\mathbf{z}_i - \mathbf{z}_j)$. Applying the control law (18), with (23) and (24), we obtain

$$\dot{V}_3 = - \sum_{i=1}^n k_i^\phi \phi_i^T \phi_i - \sum_{i=1}^n k_i^v k_i^\psi (\xi_i - \psi_i)^T (\xi_i - \psi_i), \quad (27)$$

which is negative semi-definite and we can conclude that \mathbf{z}_i , ϕ_i , for $i \in \mathcal{N}$, $(\xi_i - \xi_j)$, for $(i, j) \in \mathcal{E}$, and $(\xi_i - \psi_i)$, for $i \in \mathcal{I}$, are bounded. Since the information flow between vehicles is assumed connected, we know that $(\xi_i - \xi_j)$ is bounded, for $(i, j) \in \mathcal{N}$. To completely prove point (ii), we still need to show that the vectors θ_i and $\dot{\theta}_i$ are bounded, for $i \in \mathcal{N}$.

Using the above results, we know from (23) that $\dot{\phi}_i$ is bounded, for $i \in \mathcal{N}$, and from (24) that $\dot{\psi}_i$ is bounded, for $i \in \mathcal{I}$. As a result, \dot{V}_3 is bounded and we conclude from Barbălat Lemma that $\lim_{t \rightarrow \infty} \phi_i(t) = 0$, for $i \in \mathcal{N}$, and $\lim_{t \rightarrow \infty} (\xi_i(t) - \psi_i(t)) = 0$, for $i \in \mathcal{I}$. Using similar arguments as in the proof of Theorem 1, we can conclude that $\lim_{t \rightarrow \infty} \dot{\phi}_i(t) = 0$, for $i \in \mathcal{N}$, $\lim_{t \rightarrow \infty} (\mathbf{z}_i(t) - \dot{\psi}_i(t)) = 0$, for $i \in \mathcal{I}$. Therefore, we conclude from (23) that: $\lim_{t \rightarrow \infty} (\sum_{j=1}^n k_{ij} (\xi_i(t) - \xi_j(t))) = 0$, for $i \in \mathcal{N}$. Following similar steps as in the proof of Theorem 1, we can conclude that $\lim_{t \rightarrow \infty} (\xi_i(t) - \xi_j(t)) = 0$ and $\lim_{t \rightarrow \infty} (\mathbf{z}_i(t) - \mathbf{v}_d(t)) = 0$, for all $i, j \in \mathcal{N}$, since the communication graph is connected and $\mathcal{I} \neq \emptyset$. As a result, we have $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = \lim_{t \rightarrow \infty} (\theta_i(t) - \theta_j(t))$ and $\lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_d(t)) = \lim_{t \rightarrow \infty} \dot{\theta}_i(t)$, for all $i, j \in \mathcal{N}$.

Exploiting the above results, the dynamics of the variable θ_i can be written as in (22), with $\eta_i = (-\dot{\phi}_i + k_i^v (\xi_i - \psi_i) + \sum_{j=1}^n k_{ij} (\xi_i - \xi_j))$, which is globally bounded and converges asymptotically to zero. Then from the results of Lemma 1, we can conclude that θ_i and $\dot{\theta}_i$ are globally bounded and $\lim_{t \rightarrow \infty} \theta_i(t) = \lim_{t \rightarrow \infty} \dot{\theta}_i(t) = 0$, for $i \in \mathcal{N}$. Finally, we conclude that $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = \lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_d(t)) = 0$ for all $i, j \in \mathcal{N}$. This ends the proof. \square

Remark 5. It is important to mention that the idea behind the introduction of the new variables θ_i in the above control scheme is to modify the trajectories of the states during the transient. In fact, instead of attempting to drive the relative positions and velocity tracking errors directly to zero, we first force $(\mathbf{r}_i - \mathbf{r}_j)$ and $(\mathbf{v}_i - \mathbf{v}_d)$ to converge respectively to $(\theta_i - \theta_j)$ and $\dot{\theta}_i$ without velocity measurements. This is accomplished using the output of the first order auxiliary systems (23) and (24). Once this is achieved, the variables θ_i and $\dot{\theta}_i$ are guaranteed to converge to zero

asymptotically, for all $i \in \mathcal{N}$, guiding the relative positions and velocity tracking errors towards zero.

When it is desirable that the vehicles achieve consensus in their final states without reference velocity, i.e., $\mathbf{r}_i(t) \rightarrow \mathbf{r}_j(t)$ and $\mathbf{v}_i(t) \rightarrow \mathbf{v}_j(t)$ asymptotically, for all $i, j \in \mathcal{N}$, we propose the consensus algorithm stated in the following corollary.

Corollary 2. Consider a group of n -vehicles modeled as in (1), with the control input

$$\begin{cases} \mathbf{u}_i = -\Phi_i, \\ \ddot{\theta}_i = -\Phi_i - \dot{\phi}_i + \sum_{j=1}^n k_{ij} (\xi_i - \xi_j), \\ \dot{\phi}_i = -k_i^\phi \phi_i - \sum_{j=1}^n k_{ij} (\xi_i - \xi_j), \end{cases} \quad (28)$$

for $i \in \mathcal{N}$, with k_i^ϕ and k_{ij} defined as in Theorem 1, Φ_i is given in (19), and $\theta_i(0)$, $\dot{\theta}_i(0)$ and $\phi_i(0)$ can be selected arbitrarily. Let the controller gains satisfy

$$k_{\theta_{1i}} + k_{\theta_{2i}} \leq \mathbf{u}_{\max}, \quad \text{for } i \in \mathcal{N}, \quad (29)$$

for any $\mathbf{u}_{\max} > 0$, and let the communication graph \mathcal{G} be connected. Then,

- (i) $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$ for $i \in \mathcal{N}$,
- (ii) the signals \mathbf{v}_i , ϕ_i , θ_i , $\dot{\theta}_i$ and $(\mathbf{r}_i - \mathbf{r}_j)$ are globally bounded for all $i, j \in \mathcal{N}$,
- (iii) $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = \lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_j(t)) = 0$ for all $i, j \in \mathcal{N}$.

Proof. The result (i) is obvious from the control law (28) with (19) and condition (29). To prove points (ii) and (iii), we consider the following Lyapunov function candidate

$$\begin{aligned}
V_4 &= \frac{1}{2} \sum_{i=1}^n (\mathbf{z}_i - \phi_i)^T (\mathbf{z}_i - \phi_i) \\
&+ \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n k_{ij} (\xi_i - \xi_j)^T (\xi_i - \xi_j) + \frac{1}{2} \sum_{i=1}^n \phi_i^T \phi_i. \quad (30)
\end{aligned}$$

Following similar steps as in the proof of Theorem 2, the time derivative of V_4 evaluated along the system dynamics (1), with (28), is given by

$$\dot{V}_4 = - \sum_{i=1}^n k_i^\phi \phi_i^T \phi_i, \quad (31)$$

and we can conclude that \mathbf{z}_i , ϕ_i and $(\xi_i - \xi_j)$, for all $i, j \in \mathcal{N}$, are bounded. Using similar arguments as in the proofs of Theorems 1 and 2, we conclude that $\lim_{t \rightarrow \infty} \phi_i(t) = 0$, $\lim_{t \rightarrow \infty} \dot{\phi}_i(t) = 0$, for $i \in \mathcal{N}$, $\lim_{t \rightarrow \infty} (\xi_i(t) - \xi_j(t)) = 0$, and $\lim_{t \rightarrow \infty} (\mathbf{z}_i(t) - \mathbf{z}_j(t)) = 0$, for all $i, j \in \mathcal{N}$. Then, using the results of Lemma 1, with $\eta_i = (-\dot{\phi}_i + \sum_{j=1}^n k_{ij} (\xi_i - \xi_j))$, we conclude that θ_i and $\dot{\theta}_i$ are bounded and $\lim_{t \rightarrow \infty} (\mathbf{r}_i(t) - \mathbf{r}_j(t)) = \lim_{t \rightarrow \infty} (\mathbf{v}_i(t) - \mathbf{v}_j(t)) = 0$, for all $i, j \in \mathcal{N}$. \square

Remark 6. To implement the above control schemes, it is required that vehicles transmit their variables ξ_i . Hence, the proposed consensus algorithm does not increase the communication requirements as compared to (2).

Remark 7. The main advantage from the introduction of the auxiliary system with output θ_i in this section is to enable the design of a bounded control input for each vehicle, which does not depend explicitly on the vehicles' relative positions (as can be seen from (18) and (28) with (19)). Consequently, the upper bound of the control input of each vehicle can be determined *a priori* independently

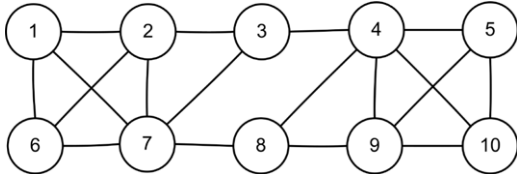


Fig. 1. Interaction graph between vehicles \mathcal{G} .

from the number of its neighbors in the group. This facilitates considerably the tuning of the controller gains to achieve good transient performance regardless of the communication topology between vehicles. In the case of the control scheme (28) for example, the gains k_i^ϕ and k_{ij} can be freely selected such that the term $\eta_i = (-\dot{\phi}_i + \sum_{j=1}^n k_{ij}(\xi_i - \xi_j))$ converges to zero as fast as possible. Thereafter, the choice of the gains $k_{\theta_{1i}}$ and $k_{\theta_{2i}}$ will determine the way \mathbf{r}_i converges to the final position for each vehicle. It is worth noting that when $\eta_i \rightarrow 0$ and θ_i has not yet converged to zero but becomes sufficiently small, we have $\dot{\mathbf{v}}_i \rightarrow \dot{\theta}_i \approx -k_{\theta_{1i}}\theta_i - k_{\theta_{2i}}\dot{\theta}_i$. This leads us to expect that at this moment of the transient response, sufficient damping is imposed on θ_i by an appropriate choice of $k_{\theta_{2i}}$. As a result, and since $(\mathbf{r}_i - \mathbf{r}_j)$ has converged to $(\theta_i - \theta_j)$, sufficient damping is implicitly applied to the system.

Remark 8. The design methodology presented in this section can be applied to extend the consensus algorithms proposed in [8], in the full information case, to achieve *a priori* boundedness of the control input with upper bounds independent of the number of neighbors.

5. Simulation results

In this section, we provide simulation results to demonstrate the effectiveness of the two control schemes proposed in this work.

We consider a group of 10 vehicles modeled as in (1), with $m = 1$, and with initial conditions: $R(0) = (1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5)^T$ and $V(0) = (0.1, 0.02, -0.08, 0.05, 0.07, -0.1, -0.05, 0.06, 0.04, 0.08)^T$, where $R(t)$ and $V(t)$ are the vectors containing respectively the positions and velocities of the vehicles, $\mathbf{r}_i(t)$ and $\mathbf{v}_i(t)$ for $i \in \{1, \dots, 10\}$. We consider the communication topology between vehicles as described by the undirected and connected graph \mathcal{G} shown in Fig. 1.

To consider the results of Theorems 1 and 2, we assume that the desired velocity is given by $\mathbf{v}_d(t) = 0.5 \sin(2t/\pi)$, and all vehicles are constrained such that $\mathbf{u}_{\max} = 2.8$. We first consider the control law (2)–(4) with, $\lambda^\phi = \lambda^\psi = k_i^\psi = 1$, $\lambda^r = 10$, $k_i^\phi = 0.5$, for $i \in \{1, \dots, 10\}$, $k_{ij} = 0.15$, for $(i, j) \in \mathcal{E}$ and $k_i^\nu = 0.2$ for $i \in \mathcal{L} = \{1, 7\}$, i.e., the auxiliary system (4) is implemented for only vehicles 1 and 7. Note that this choice of gains with the upper bound of $\dot{\mathbf{v}}_d(t)$ satisfy condition (5). The results are illustrated in Fig. 2. We can see that all vehicles reach consensus on their final positions with the final desired velocity after 35 s.

Then, we consider control law (18) and (23)–(24), with $k_{\theta_{1i}} = 0.7$, $k_{\theta_{2i}} = 1.1$, $k_i^\phi = 10$, $k_i^\psi = 1$, for $i \in \{1, \dots, 10\}$, $k_{ij} = 100$, for $(i, j) \in \mathcal{E}$, and $k_i^\nu = 5$ for $i \in \mathcal{L} = \{1, 7\}$, i.e., the auxiliary system (24) is implemented in only vehicles 1 and 7. Note that the choice of the gains $k_{\theta_{1i}}$ and $k_{\theta_{2i}}$ with the upper bound of $\dot{\mathbf{v}}_d(t)$ satisfy condition (21), with $\mathbf{u}_{\max} = 2.8$. The results are illustrated in Fig. 3, where we can see that faster consensus with final desired velocity is achieved using this consensus algorithm.

From these figures, we notice the oscillations present in the system response and the control input of each vehicle. This is mainly due to the “insufficient” damping introduced to the system by the auxiliary variables used to substitute the missing velocity information. However, we can see that in the second approach, Fig. 3, oscillations in the system response and in the control input of each vehicle decay rapidly after few seconds, whereas in the first approach, Fig. 2, oscillations are present for a longer time. It

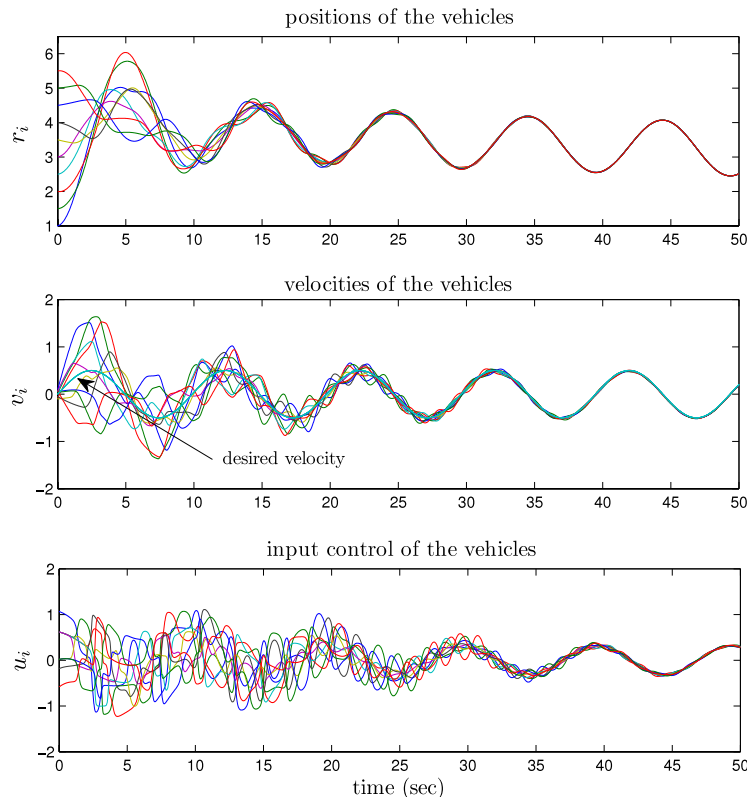


Fig. 2. Results of Theorem 1, with $\mathbf{u}_{\max} = 2.8$.

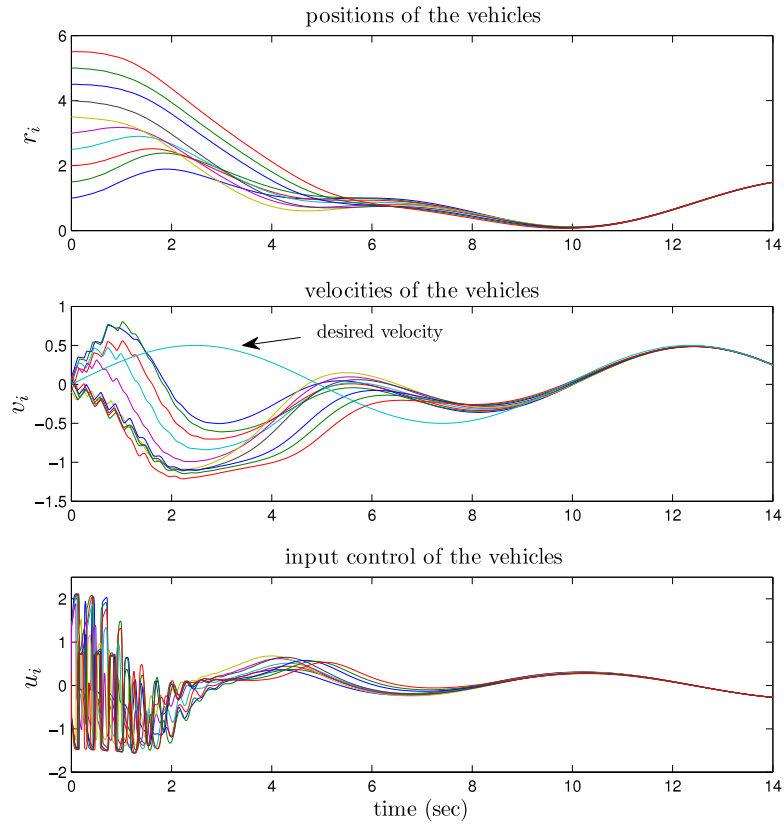


Fig. 3. Results of Theorem 2, with $u_{\max} = 2.8$.

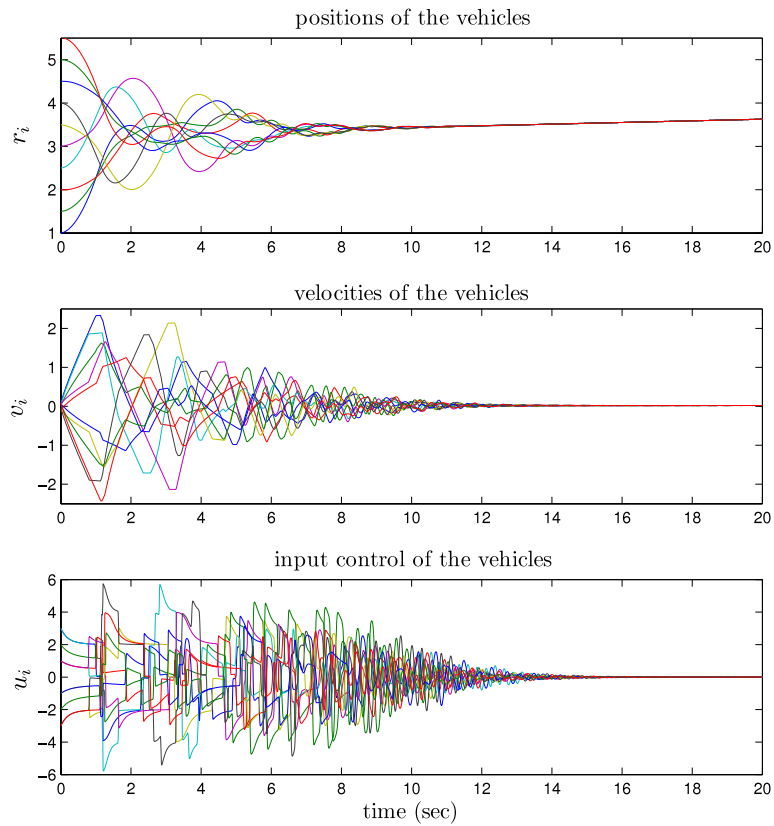


Fig. 4. Results of Corollary 1, with $u_{\max} = 6$.

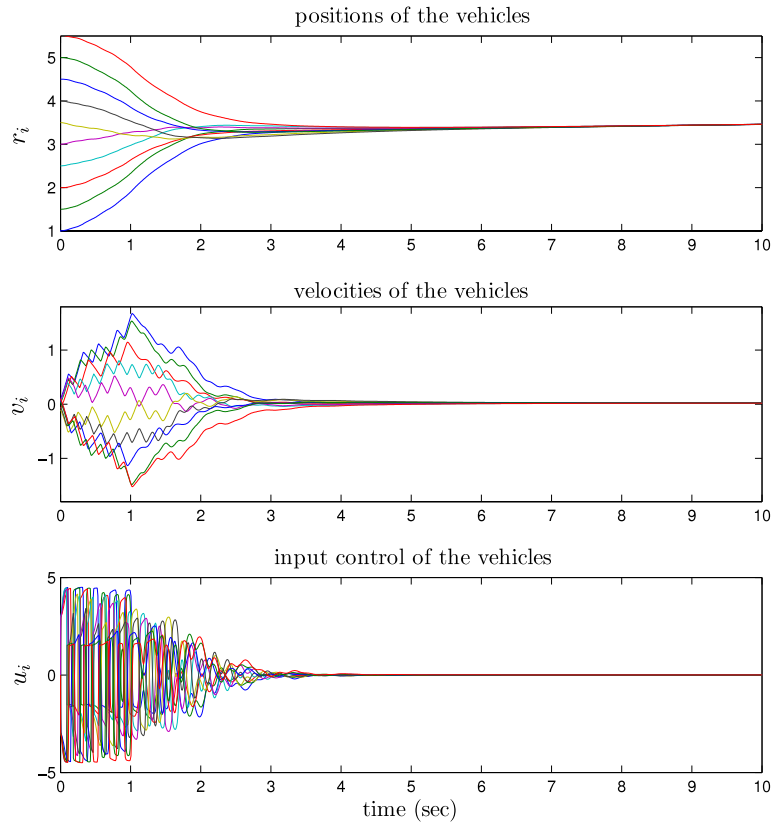


Fig. 5. Results of Corollary 2, with $\mathbf{u}_{\max} = 6$.

is interesting to investigate the response of the system when the second approach is implemented. From the control input in Fig. 3, we can see the high control activity for each vehicle in the first period of time, (0, 1) s, where the maximum input, as determined in (20), is almost reached. This indicates that the variables θ_i , $\dot{\theta}_i$ and $\eta_i = (-\dot{\phi}_i + k_i^\psi(\xi_i - \psi_i) + \sum_{j=1}^n k_{ij}(\xi_i - \xi_j))$ have not yet reached small values. Note that this is due to the high values of the gains k_i^ϕ , k_{ij} and k_i^ψ , which are selected independently from the system constraints and aim to drive the term η_i to zero as fast as possible. After the instant 1 s, we see from the same figure that the value of the control input for each vehicle starts decreasing. This indicates that the variables θ_i and $\dot{\theta}_i$ are decreasing, and the term η_i is approaching zero. At this point, the dynamics of $(\mathbf{r}_i - \mathbf{r}_j)$ are approaching the dynamics of $(\theta_i - \theta_j)$, for $(i, j) \in \mathcal{E}$, and with the above choice of $k_{\theta_{1i}}$ and $k_{\theta_{2i}}$, we are adding damping to the response of θ_i , and consequently damping is introduced to the response of \mathbf{r}_i , for $i \in \{1, \dots, 10\}$. As a result, the transient oscillations in the system response are considerably reduced after few seconds.

Next, we consider the results of Corollary 1. We assume that no reference velocity is assigned to the team and all vehicles are constrained such that $\mathbf{u}_{\max} = 6$. Fig. 4 illustrates the obtained results when control law (13) is implemented with $\lambda^\phi = 3$, $\lambda^r = 50$, $k_i^\phi = 1$, for $i \in \{1, \dots, 10\}$, and $k_{ij} = 0.5$, for $(i, j) \in \mathcal{E}$. Note that this choice of the control gains satisfies condition (14), and it achieves the fastest consensus that we have obtained with (13). We can see that consensus is achieved after 12 s with the presence of some oscillations in the vehicles' positions and velocities. To investigate the effects of the control upper bound on the system response, we consider the same control law, (13), with $\mathbf{u}_{\max} = 2$, $\lambda^\phi = 3$, $\lambda^r = 100$, $k_i^\phi = 0.4$, for $i \in \{1, \dots, 10\}$, and $k_{ij} = 0.16$, for $(i, j) \in \mathcal{E}$. The obtained results are illustrated in Fig. 6, where we can see that consensus is achieved after more than 25 s. It is important to mention that with the same information flow

between vehicles, and smaller values of \mathbf{u}_{\max} , tuning the controller gains to achieve good results becomes more difficult.

Finally, we consider the results of Corollary 2 with $\mathbf{u}_{\max} = 6$. In Fig. 5, we show the obtained results when the control scheme in (28) is implemented with $k_i^\phi = 10$, $k_{\theta_{1i}} = 1.5$ and $k_{\theta_{2i}} = 3$, for $i \in \{1, \dots, 10\}$, and $k_{ij} = 100$, for $(i, j) \in \mathcal{E}$. Note that this choice of gains satisfy condition (29). Due to the controller gains tuning flexibility introduced by this method, and as discussed above, we can see that the transient oscillations are reduced and faster consensus is achieved as compared to the results in Fig. 4. In the case where $\mathbf{u}_{\max} = 2$, we consider the same control law, (28) with $k_{\theta_{1i}} = 0.65$ and $k_{\theta_{2i}} = 1.35$, for $i \in \{1, \dots, 10\}$, with the same previous gains, and the obtained results are shown in Fig. 7. We can notice that similar performance is achieved, as compared to Fig. 5, even with lower upper bounds of the control efforts for each vehicle.

6. Conclusion

We considered the consensus problem for double-integrator dynamics without velocity measurements and with input constraints, under fixed and undirected information flow. First, based on the auxiliary systems approach, we proposed a velocity-free consensus scheme which extends the work in [8] to account for actuator saturations. Although consensus is attained, the proposed scheme presents some limitations in the sense that the upper bound of the control input for each vehicle depends on the number of its neighbors. To overcome this problem, we proposed a second velocity-free consensus algorithm with *a priori* bounded control inputs, where the upper bounds are not affected by the number of neighbors of each vehicle. The main idea in this approach is the introduction of an additional second order auxiliary system in terms of θ_i for each vehicle. As a result, additional damping is generated through the dynamics of the new variables leading

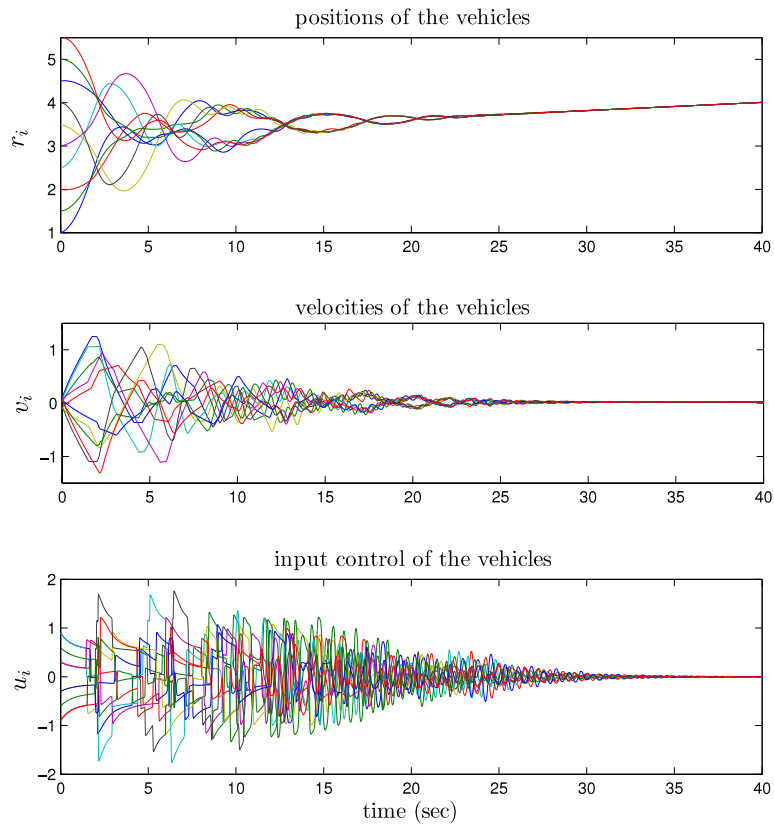


Fig. 6. Results of Corollary 1, with $u_{\max} = 2$.

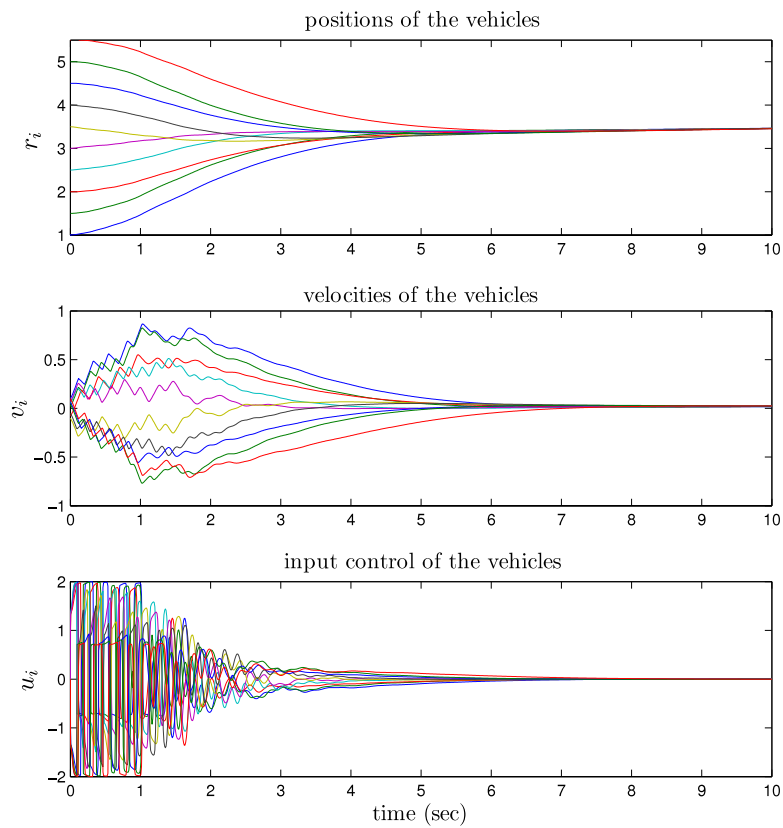


Fig. 7. Results of Corollary 2, with $u_{\max} = 2$.

to reduce the transient oscillations of the vehicles' positions before reaching consensus. In contrast to the existing velocity-free consensus algorithms, an important feature of the proposed schemes in this paper is that they can handle time-varying trajectories. Our future work will consider the extension of the proposed consensus algorithms in the presence of communication delays and in the case of directed and/or switching communication topologies.

Appendix

Proof of Lemma 1. Consider the following Lyapunov function candidate

$$W_i = \frac{1}{2} \dot{\theta}_i^T \dot{\theta}_i + k_{\theta_{2i}} \mathbf{1}_m^T \log(\cosh(\theta_i)), \quad (\text{A.1})$$

The time derivative of W_i along the dynamics (22) is given by

$$\begin{aligned} \dot{W}_i &= -\dot{\theta}_i^T (k_{\theta_{2i}} \tanh(\dot{\theta}_i) - \eta_i) \\ &\leq -\sum_{k=1}^m |\dot{\theta}_i^k| (k_{\theta_{2i}} \tanh(|\dot{\theta}_i^k|) - |\eta_i^k|), \end{aligned} \quad (\text{A.2})$$

where $\theta_i = \text{col}[\theta_i^k]$ and $\eta_i = \text{col}[\eta_i^k]$, for $k = 1 \dots m$, and we have used the property $x \tanh(x) = |x| \tanh(|x|)$ for any $x \in \mathbb{R}$. First of all, let us show that θ_i and $\dot{\theta}_i$ cannot escape in finite time. In fact, from (A.2) it is clear that $\dot{W}_i \leq \|\dot{\theta}_i\| \|\eta_i\|$. Using the fact that $\|\dot{\theta}_i\|^2 \leq 2W_i$, we have $\dot{W}_i \leq \bar{\eta}_i \sqrt{W_i}$, with $\sqrt{2} \|\eta_i\| \leq \bar{\eta}_i$, which can be rewritten as

$$\frac{dW_i}{\sqrt{W_i}} \leq \bar{\eta}_i dt. \quad (\text{A.3})$$

Integrating the last inequality over the interval $[t_0, t]$ yields

$$2 \left(\sqrt{W_i(t)} - \sqrt{W_i(t_0)} \right) \leq \bar{\eta}_i (t - t_0), \quad (\text{A.4})$$

which shows that $W_i(t)$ cannot go to infinity in finite time.

Now, we will show the global boundedness and convergence of θ_i and $\dot{\theta}_i$ to zero. It is easily seen that the right hand side of (A.2) is negative as long as

$$\tanh(|\dot{\theta}_i^k|) > \frac{|\eta_i^k|}{k_{\theta_{2i}}}, \quad \text{for } k = 1 \dots m. \quad (\text{A.5})$$

Due to the definition of \tanh , inequality (A.5) cannot be satisfied when $|\eta_i^k| > k_{\theta_{2i}}$, for $k = 1 \dots m$. However, since $\eta_i(t)$ is bounded and converges asymptotically to zero, it is clear that there exists a finite time t_1 such that $|\eta_i^k(t)| \leq k_{\theta_{2i}}$ for all $t \geq t_1$. Note that θ_i and $\dot{\theta}_i$ remain bounded on the interval $[0, t_1]$ as there is no

finite escape time. Consequently, for all $t \geq t_1$, one can conclude that $\dot{W}_i < 0$, and θ_i and $\dot{\theta}_i$ are bounded outside the set $\mathcal{M} = \left\{ \dot{\theta}_i \mid \tanh(|\dot{\theta}_i^k|) \leq \frac{|\eta_i^k|}{k_{\theta_{2i}}}, \text{ for } k = 1 \dots m \right\}$. From the properties of the function $\tanh(\cdot)$, we conclude that $\dot{\theta}_i$ is ultimately bound to reach the set \mathcal{M} and will be driven to zero as $\lim_{t \rightarrow \infty} \eta_i(t) = 0$. Finally, using a special version of Barbälát Lemma (see, for instance, Lemma 2 in [18]), together with the fact that η_i and θ_i are bounded and converge asymptotically to zero, we can show that $\lim_{t \rightarrow \infty} \theta_i(t) = 0$.

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