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Synchronization of multiple Euler-Lagrange systems with communication delays

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Abstract— **The synchronization problem of networked Euler-Lagrange systems is addressed. The information flow in the network is subject to unknown time-varying (possibly discontinuous) communication delays, and is represented by a directed communication graph that contains a spanning tree. Using a small gain framework, it is shown that the proposed control scheme achieves synchronization under mild and realistic assumptions on the communication delays. Simulation results are presented that confirm the validity of the proposed approach.**

I. Introduction

Recently, the synchronization problem of nonlinear systems has received a growing interest in the control community. In particular, mechanical systems modeled by Euler-Lagrange equations have been the focus of several research groups and several results have been reported in the literature with applications involving spacecraft formations [1] and robot networks [2], [3], [4]. The main control objective is to design appropriate input laws such that the networked systems synchronize their states using local information exchange – An information exchange that is generally restricted and is subject to delays inherent to communication systems.

In the case of constant communication delays, the authors in [5] have shown that output synchronization of nonlinear passive systems can be achieved provided that the interconnection topology between the systems is represented by a balanced and strongly connected graph. The passivity property of the systems has also been exploited in [6] and [7], where synchronization schemes with trajectory tracking have been developed for multiple Euler-Lagrange systems. In [8], the authors presented synchronization schemes for nonlinear systems of relative degree two under undirected communication graphs. With the same assumption on the interconnection graph, the authors in [9] proposed synchronization schemes for multiple Euler-Lagrange systems that account for input saturations. More recently, the assumption on the interconnection topology has been relaxed in [10], where control laws that achieve position synchronization under directed connected communication graphs have been proposed. However, the assumption of constant communication delays is not generally satisfied in practical implementations. In fact, communication over networks imposes restrictive constraints that include unknown, time-varying, and possibly discontinuous communication delays.

The effects of time-varying communication delays are generally studied using Lyapunov-Krasovskii functionals, where the main objective is to derive sufficient conditions on the control gains and the upper bounds of the delay functions such that the control goal is achieved. Based on this method, some interesting results on the synchronization of networked nonlinear systems have been proposed in [11] in the case of unmanned aerial vehicles, and in [12] in the case of spacecraft formations. As witnessed in these papers, the use of Lyapunov-Krasovskii functionals for this type of nonlinear systems requires an undirected communication topology, in addition to the differentiability and/or known upper bounds of the time-varying communication delays.

In this paper, we present a solution to the synchronization problem of Euler-Lagrange systems in the presence of time-varying communication delays. The interconnection topology between the systems in the network is represented by a directed communication graph. The effects of communication delays are addressed using the multi-dimensional small gain approach for interconnected systems with communication delays. This approach is based on the notion of input-to-state stability (ISS) [13] and the version of the input-to-output stability (IOS) small-gain theorem for interconnected systems with constrained communication presented in [14]. Using this approach, we show that synchronization in the presence of time-varying communication delays can be achieved with an appropriate choice of the control gains provided that the directed communication graph contains a spanning tree. It is worth emphasizing that the approach imposes a mild assumption on the communication process which, in particular, allows for bounded discontinuous communication delay functions with unknown upper bounds. Simulation results on a network of ten robot manipulators are given to illustrate the performance of the obtained results.

The paper is organized as follows. Section II is devoted to the problem formulation, and Section III presents some definitions and preliminary technical results. Section IV contains the formulation and the proof of our main result. Simulation results are presented in Section V, while in Section VI some concluding remarks are given.

II. Problem Statement

Consider a network of n not necessarily identical systems governed by the Euler-Lagrange equations of the form

$$\mathbf{M}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i + \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i + \mathbf{G}_i(\mathbf{q}_i) = \mathbf{u}_i, \qquad (1)$$

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for $i \in \mathcal{N} \triangleq \{1,...,n\}$, with $\mathbf{q}_i \in \mathbb{R}^m$ is the vector of generalized configuration coordinates, $\mathbf{M}_i(\mathbf{q}_i) \in \mathbb{R}^{m\times m}$ is the positive-definite inertia matrix, $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i$ is the vector of coriolis/centrifugal forces, $\mathbf{G}_i(\mathbf{q}_i)$ is the vector of gravitational force, and $\mathbf{u}_i$ is the vector of torques associated with the i^{th} system. Also, we assume that the coriolis matrix is defined such that $\dot{\mathbf{M}}_i = \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) + \mathbf{C}_i^\top(\mathbf{q}_i, \dot{\mathbf{q}}_i)$.

The information flow between the Euler-Lagrange systems is described by the directed interconnection graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{K})$. The set $\mathcal{N}$ is the set of nodes or vertices, describing the set of systems in the network, $\mathcal{E} \in \mathcal{N}\times\mathcal{N}$ is the set of ordered pairs of nodes, called edges, and $\mathcal{K} = [k_{ij}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix. An edge (i, j) indicates that system j can receive information from system i, but not necessarily vice versa. The weighted adjacency matrix is defined such that $k_{ii} \triangleq 0$, $k_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $k_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), ...$, where $i_l \in \mathcal{N}$. A directed graph is said to contain a directed spanning tree if there exists at least one node having a directed path to all of the other nodes.

The Laplacian matrix $\mathcal{L} := [l_{ij}] \in \mathbb{R}^{n\times n}$ of the directed graph $\mathcal{G}$ is defined such that: $l_{ii} = \sum_{j=1}^{n} k_{ij}$, and $l_{ij} = -k_{ij}$ for $i \neq j$. In view of its definition, the Laplacian matrix satisfies $\mathcal{L}\mathbf{1}_n = 0$, with $\mathbf{1}_n \in \mathbb{R}^n$ is the vector with every element equal to one. Moreover, if the directed graph has a directed spanning tree then $\mathcal{L}$ has a single zero-eigenvalue and the rest of the spectrum of $\mathcal{L}$ has positive real parts [15].

We assume that each system can sense its state vector with no delay, and for any pair of nodes $(j, i) \in \mathcal{E}$, the information of j^{th} system is received by the i^{th} system with the communication delay $\tau_{ij}(t)$. The following assumption is imposed on the communication delays $\tau_{ij}(t)$.

Assumption 1: For each $(j, i) \in \mathcal{E}$, the communication delay $\tau_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ can be decomposed into the sum of two terms,

$$\tau_{ij}(t) = \tau_{ij}^s(t) + \tau_{ij}^r(t), \tag{2}$$

where the components $\tau_{ij}^s(\cdot)$ and $\tau_{ij}^r(\cdot)$ have the following properties:

i) There exists $\Upsilon_{ij} \geq 0$ such that the inequality

$$\left|\tau_{ij}^s(t_2) - \tau_{ij}^s(t_1)\right| \leq \Upsilon_{ij} \cdot |t_2 - t_1|, \tag{3}$$

holds for almost all $t_2, t_1 \in \mathbb{R}_+$, with $t_2 \geq t_1$.

ii) The function $\tau_{ij}^s(t)$ satisfies:

$$t - \tau_{ij}^s(t) \to +\infty \quad \text{as } t \to +\infty, \tag{4}$$

iii) There exists a function $\tau^* : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\tau^*(t_2) - \tau^*(t_1) \leq t_2 - t_1 \tag{5}$$

for $t_2 \geq t_1$, and

$$\left|\tau_{ij}^s\right| \leq \tau^*(t) \tag{6}$$

holds for all $t \geq 0$.

iv) There exists $\Delta_{ij}^\tau \geq 0$ such that

$$\left|\tau_{ij}^r(t)\right| \leq \Delta_{ij}^\tau \tag{7}$$

holds for almost all $t \geq 0$.

The subscripts s and r indicate that $\tau_{ij}^s(\cdot)$ and $\tau_{ij}^r(\cdot)$ are the "smooth" and the "random" components of the communication delay, respectively. In particular, the condition (3) implies that, for each $(j, i) \in \mathcal{E}$, the time derivative $d\tau_{ij}^s(t)/dt$ is well-defined for almost all $t \geq 0$ and satisfies $\left|d\tau_{ij}^s(t)/dt\right| \leq \Upsilon_{ij}$, where defined. Also, (5)-(6) imply the existence of an upper bound τ^*, which is possibly a time-varying unbounded function of time that does not grow faster than the time itself.

With the above definitions and assumptions, our objective is to design a control scheme for each system such that all systems synchronize their positions with zero final velocity *i.e.,* $\dot{\mathbf{q}}_i \to 0$, and $(\mathbf{q}_i - \mathbf{q}_j) \to 0$, for $i, j \in \mathcal{N}$.

III. Preliminary results

In this section we present some definitions and preliminary results that will be used in the subsequent analysis. We start from the following simple lemma (which can be found, for example, in [16], [17]).

Lemma 1: Consider an LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{8}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the input vector, and $\mathbf{A}$ is a Hurwitz matrix such that $\mathbf{A} + \mathbf{A}^* + 2\nu\mathbf{I} \leq 0$ for some $\nu > 0$. Then for any initial condition $\mathbf{x}(t_0)$, the solution of (8) satisfies

$$|\mathbf{x}(t)| \leq e^{-\nu(t-t_0)}|\mathbf{x}(t_0)| + \frac{\|\mathbf{B}\|}{\nu} \sup_{\sigma\in[t_0,t]} |\mathbf{u}(\sigma)|. \tag{9}$$

Next, we present the small-gain theorem that will be used in the proof of our main results. Consider an affine nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{10}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are locally Lipschitz functions of the corresponding dimensions, $f(0) = 0$, $h(0) = 0$. In the definition below, we need the notion of $\mathcal{K}$ and $\mathcal{K}_\infty$ functional classes defined as follows. A continuous function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ ($\gamma \in \mathcal{K}$) if it is strictly increasing and satisfies $\gamma(0) = 0$. A function $\gamma \in \mathcal{K}$ belongs to class $\mathcal{K}_\infty$ if $\gamma(s) \to \infty$ as $s \to \infty$. The following notion of input-to-state stability was introduced by Sontag (see, for example, [13]).

Definition 1: A system of the form (10) is said to be *input-to-state stable* (ISS) if there exist $\beta \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that the following inequalities hold along the trajectories of the system:

i) *uniform boundedness:*

$$|x(t)| \leq \max\left\{\beta\left(|x(0)|\right), \gamma\left(\sup_{s\in[0,t)} |u(s)|\right)\right\}$$

holds for all $t \geq 0$, and

ii) *asymptotic gain:*

$$\limsup_{t\to+\infty} |x(t)| \leq \limsup_{t\to+\infty} \gamma\left(|u(t)|\right).$$

In the above definition, $\gamma \in \mathcal{K}$ is called the ISS gain. It is worth mentioning that for a system of the form (10), the input-to-state stability implies the so called input-to-output stability, which means that there exist $\beta_y, \gamma_y \in \mathcal{K}_\infty$ such that

$$|y(t)| \leq \max\left\{\beta_y\left(|x(0)|\right), \gamma_y\left(\sup_{s\in[0,t)}|u(s)|\right)\right\}$$

holds for all $t \geq 0$, and also

$$\limsup_{t\to+\infty}|y(t)| \leq \limsup_{t\to+\infty}\gamma_y\left(|u(t)|\right).$$

In this case, $\gamma_y \in \mathcal{K}_\infty$ is called the IOS (input-to-output stability) gain. In addition, the ISS (IOS) gain $\gamma \in \mathcal{K}_\infty$ can be a linear function, $\gamma(s) := \gamma^0 \cdot s$, where $\gamma^0 \geq 0$. In this case, we will say that the system (10) has linear ISS (IOS) gain.

The following small-gain theorem is the key technical tool used in our work.

Theorem 1: Consider n affine subsystems

$$\begin{array}{rcl} \dot{x}_i & = & f_i(x_i) + g_i(x_i)u_i \\ y_i & = & h_i(x_i), \end{array} \tag{11}$$

where $i = 1, \ldots, n$. Suppose each subsystem (11) is ISS and the corresponding IOS gain γ_i is linear with $\gamma_i^0 > 0$. Suppose also that each input $u_i(t)$ is Lebesgue measurable function satisfying: $u_i(t) \equiv 0$ for $t < 0$, and there exist $\mu_{ij} \geq 0$, $i, j \in \{1, \ldots, n\}$, such that the following inequalities hold

$$|u_i(t)| \leq \sum_{j\in\{1,\ldots,n\}} \mu_{ij} \cdot \left(\sup_{s\in[t-\tau_{ij}(t),t]}|y_j(s)|\right), \tag{12}$$

where all $\tau_{ij}(t)$ satisfy Assumption 1. Let

$$\Gamma := \Gamma^0 \cdot \mathcal{M} \in \mathbb{R}^{n\times n},$$

where $\Gamma^0 := \text{diag}\left\{\gamma_i^0\right\}$, $\mathcal{M} := \{\mu_{ij}\}$, $i, j \in \{1, \ldots, n\}$. If

$$\rho\left(\Gamma\right) < 1, \tag{13}$$

where $\rho\left(\Gamma\right)$ is the spectral radius of the matrix Γ, then the trajectories of the system (11) with input-output constraints (12) are uniformly bounded and convergent.

The above theorem is "almost" a special case of Theorem 1 in [14], and can be proven by combination of Theorem 1 in [14] and Corollary 16 from [18]. The proof is omitted.

IV. Main result

We propose the following control algorithm for each system

$$\mathbf{u}_i = -\mathbf{M}_i(\mathbf{q}_i)\Lambda_i\dot{\tilde{\mathbf{q}}}_i - \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\Lambda_i\tilde{\mathbf{q}}_i + \mathbf{G}_i(\mathbf{q}_i) - k_i^s\mathbf{s}_i, \tag{14}$$

where $\Lambda_i = \Lambda_i^T > 0$ is a symmetric positive definite matrix, k_i^s is a positive scalar gain, and

$$\tilde{\mathbf{q}}_i = \kappa_i\mathbf{q}_i - \psi_i^{\{1\}}, \tag{15}$$

with $\psi_i^{\{1\}}$ being the output of the following filter

$$\left\{\begin{array}{rcl} \dot{\psi}_i^{\{1\}} & = & \psi_i^{\{2\}} \\ \dot{\psi}_i^{\{2\}} & = & -\alpha_1\psi_i^{\{2\}} - \alpha_0\psi_i^{\{1\}} \\ & & +\alpha_0\sum_{j=1}^n k_{ij}\mathbf{q}_j(t-\tau_{ij}(t)) \end{array}\right., \tag{16}$$

where $\psi_i^{\{1\}}(0)$, $\psi_i^{\{2\}}(0)$ can be selected arbitrarily, α_1, $\alpha_0 > 0$, k_{ij} is the (i,j)-th element of the adjacency matrix $\mathcal{K}$ of the directed communication graph $\mathcal{G}$, and $\kappa_i := \sum_{j=1}^n k_{ij}$. The vector $\mathbf{s}_i$ in (14) is the generalized error of the i-th subsystem and is defined as follows,

$$\mathbf{s}_i = \dot{\mathbf{q}}_i + \Lambda_i\tilde{\mathbf{q}}_i. \tag{17}$$

Now, denote

$$\mu := -\max\left\{\mathcal{R}e(\mu_1), \mathcal{R}e(\mu_2)\right\}, \tag{18}$$

where μ_1, μ_2 are the roots of $p^2 + \alpha_1 p + \alpha_0 = 0$. Also, let $\lambda_{i_{\max}} \geq \lambda_{i_{\min}} > 0$ be the maximal and minimal eigenvalues, respectively, of the matrices Λ_i, $i \in \mathcal{N}$. Our main result is the following theorem.

Theorem 2: Consider the network of n-systems described by (1), where the interconnection between the subsystems is described by the directed graph $\mathcal{G}$. Let the controller be defined by (14)-(17) and suppose that Assumption 1 holds. If

$$\Gamma_i := \frac{\lambda_{i_{\max}}}{\lambda_{i_{\min}}\cdot\kappa_i\cdot\mu}\cdot\sum_{j=1}^n k_{ij}\left(1 + \Upsilon_{ij} + \alpha_0\cdot\Delta_{ij}^\tau\right) < 1, \tag{19}$$

for each $i \in \mathcal{N}$ with $\kappa_i \neq 0$, then the trajectories of the closed-loop system (1), (14)-(17) are bounded and satisfy $\dot{\mathbf{q}}_i(t) \to 0$ as $t \to +\infty$, for all $i \in \mathcal{N}$. Furthermore, if the directed communication graph $\mathcal{G}$ contains a directed spanning tree, and $\tau^*(t)$ in Assumption 1, part iii), satisfies $\limsup_{t\to+\infty}\tau^*(t) < \infty$, then $(\mathbf{q}_i(t) - \mathbf{q}_j(t)) \to 0$ as $t \to +\infty$, for all $i, j \in \mathcal{N}$.

Remark 1: It is worth pointing out that the small-gain conditions (19) impose mild constraints on the characteristics of the communication delays. To clarify these constraints, note that $\mu > 0$ can be assigned arbitrarily large by an appropriate choice of the coefficients $\alpha_1, \alpha_0 > 0$. The ratio $\lambda_{i_{\max}}/\lambda_{i_{\min}} \geq 1$ can also be assigned arbitrary. This implies that, for any given $\Upsilon_{ij} \geq 0$, $i, j \in \{1, \ldots, N\}$, the conditions (19) can always be met if $\Delta_{ij}^\tau \geq 0$, $i, j \in \{1, \ldots, n\}$, are sufficiently small. Essentially, the synchronization can be achieved for communication delays with arbitrary smooth components (more precisely, for arbitrary smooth components with given Lipschitz constant), as long as the discontinuous components are uniformly bounded with some sufficiently small positive bounds that are determined by the Lipschitz constants of the corresponding "smooth" components.

Proof: Let us first show that the error vector $\mathbf{s}_i$ is bounded and converges exponentially to zero. Taking the time-derivative of (17), we can write $\dot{\mathbf{s}}_i = \ddot{\mathbf{q}}_i + \Lambda_i\dot{\tilde{\mathbf{q}}}_i$. This, with (1) lead us to write the error dynamics as

$$\mathbf{M}_i(\mathbf{q}_i)\dot{\mathbf{s}}_i = \mathbf{u}_i - \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i - \mathbf{G}_i(\mathbf{q}_i) + \mathbf{M}_i(\mathbf{q}_i)\Lambda_i\dot{\tilde{\mathbf{q}}}_i. \tag{20}$$

Substituting the control algorithm (14) into the last equation, one gets

$$\mathbf{M}_i(\mathbf{q}_i)\dot{\mathbf{s}}_i + \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\mathbf{s}_i + k_d^s\mathbf{s}_i = 0. \tag{21}$$

Then, the time-derivative of the Lyapunov function candidate $V = \frac{1}{2}\sum_{i=1}^{n} \mathbf{s}_i^T \mathbf{M}_i(\mathbf{q}_i)\mathbf{s}_i$, is obtained as $\dot{V} = -\sum_{i=1}^{n} k_i^d \mathbf{s}_i^T \mathbf{s}_i$, which leads us to conclude that $\mathbf{s}_i$ is bounded and converges exponentially to zero.

Now, we consider all systems $i \in \mathcal{N}$ with $\kappa_i \neq 0$. It should be noted that $\kappa_i = 0$ implies that the i^{th} system does not receive information from any other system in the network. For our purposes, we introduce the following notation,

$$\hat{\mathbf{q}}_i(t) := \sum_{j=1}^{n} k_{ij}\mathbf{q}_j(t - \tau_{ij}(t)). \tag{22}$$

Clearly, $\hat{\mathbf{q}}_i$ is a linear combination of the position variables $\mathbf{q}_j$ delayed by the corresponding communication delays $\tau_{ij}(\cdot)$. Furthermore, we define the following vectors

$$\begin{aligned} \hat{\mathbf{q}}_i^s(t) &:= \sum_{j=1}^{n} k_{ij}\mathbf{q}_j\left(t - \tau_{ij}^s(t)\right), && (23)\\ \Delta\hat{\mathbf{q}}_i(t) &:= \hat{\mathbf{q}}_i(t) - \hat{\mathbf{q}}_i^s(t), && (24)\\ \tilde{\psi}_i &:= \psi_i^{\{1\}} - \hat{\mathbf{q}}_i^s, && (25) \end{aligned}$$

with $\tau_{ij}^s(t)$ being given in Assumption 1. With these definitions, the equations of the filter (16) can be rewritten as follows:

$$\begin{pmatrix} \dot{\tilde{\psi}}_i \\ \dot{\psi}_i^{\{2\}} \end{pmatrix} = \mathbf{A}\begin{pmatrix} \tilde{\psi}_i \\ \psi_i^{\{2\}} \end{pmatrix} + \mathbf{B}\begin{pmatrix} \dot{\hat{\mathbf{q}}}_i^s \\ \Delta\hat{\mathbf{q}}_i \end{pmatrix} \tag{26}$$

where $\mathbf{A} = \begin{pmatrix} \mathbf{0}_m & \mathbf{I}_m \\ -\alpha_0\mathbf{I}_m & -\alpha_1\mathbf{I}_m \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} -\mathbf{I}_m & \mathbf{0}_m \\ \mathbf{0}_m & \alpha_0\mathbf{I}_m \end{pmatrix}$.

Applying Lemma 1 to the trajectories of the filter (26), we see that the following estimate

$$\begin{aligned} \left|\begin{matrix} \tilde{\psi}_i(t) \\ \psi_i^{\{2\}}(t) \end{matrix}\right| &\leq e^{-\mu(t-t_0)} \cdot \left|\begin{matrix} \tilde{\psi}_i(t_0) \\ \psi_i^{\{2\}}(t_0) \end{matrix}\right| \\ &+ \frac{1}{\mu}\left(\sup_{\sigma\in[t_0,t]}\left|\dot{\hat{\mathbf{q}}}_i^s(\sigma)\right| + \alpha_0 \sup_{\sigma\in[t_0,t]}\left|\Delta\hat{\mathbf{q}}_i(\sigma)\right|\right), \end{aligned} \tag{27}$$

holds for any $t \geq t_0$, where $\mu > 0$ is defined by (18). Inequality (27) implies, in particular, that the filter (26) is ISS (Definition 1).

Also, the time-derivative of (15), in view of (16) and (17), gives

$$\dot{\tilde{\mathbf{q}}}_i = -\kappa_i\Lambda_i\tilde{\mathbf{q}}_i - \kappa_i\mathbf{s}_i - \psi_i^{\{2\}}. \tag{28}$$

This is a linear stable system with inputs $\mathbf{s}_i$ and $\psi_i^{\{2\}}$. Applying Lemma 1 to (28) yields us to write

$$\begin{aligned} |\tilde{\mathbf{q}}_i(t)| \leq & e^{-\kappa_i\cdot\lambda_{i_{\min}}(t-t_0)}|\tilde{\mathbf{q}}_i(t_0)| + \frac{1}{\lambda_{i_{\min}}}\cdot\sup_{\sigma\in[t_0,t]}|\mathbf{s}_i(\sigma)| \\ &+ \frac{1}{\kappa_i\cdot\lambda_{i_{\min}}}\cdot\sup_{\sigma\in[t_0,t]}\left|\psi_i^{\{2\}}(\sigma)\right|, \end{aligned} \tag{29}$$

from which we can see that $1/(\kappa_i\cdot\lambda_{i_{\min}})$ is the ISS gain of system (28) with respect to the input $\psi_i^{\{2\}}$.

We can see from (21), (26) and (28) that the closed loop system can be viewed as the interconnection of subsystems with the following dynamics

$$\begin{aligned} \dot{\mathbf{s}}_i &= \mathbf{M}_i^{-1}(\mathbf{q}_i)\left(-\mathbf{C}_i(\mathbf{q}_i,\dot{\mathbf{q}}_i)\mathbf{s}_i - k_d^s\mathbf{s}_i\right)\\ \dot{\tilde{\mathbf{q}}}_i &= -\kappa_i\Lambda_i\tilde{\mathbf{q}}_i - \kappa_i\mathbf{s}_i - \psi_i^{\{2\}}\\ \dot{\tilde{\psi}}_i &= \psi_i^{\{2\}} - \dot{\hat{\mathbf{q}}}_i^s\\ \dot{\psi}_i^{\{2\}} &= -\alpha_0\tilde{\psi}_i - \alpha_1\psi_i^{\{2\}} + \alpha_0\Delta\hat{\mathbf{q}}_i \end{aligned} \tag{30}$$

where the vectors $\mathbf{s}_i$, $\tilde{\psi}_i$, $\psi_i^{\{2\}}$ and $\tilde{\mathbf{q}}_i$ are the states, $\dot{\hat{\mathbf{q}}}_i^s$ and $\Delta\hat{\mathbf{q}}_i$ are the inputs, and the output is given by

$$\dot{\mathbf{q}}_i = \mathbf{s}_i - \Lambda_i\tilde{\mathbf{q}}_i. \tag{31}$$

Taking into account (27), (29), as well as the exponential stability of (21), we see that each subsystem (30) is ISS. Moreover, (31) implies

$$|\dot{\mathbf{q}}_i(t)| \leq \lambda_{i_{\max}}\cdot|\tilde{\mathbf{q}}_i(t)| + |\mathbf{s}_i(t)|. \tag{32}$$

Combining (27), (29) and (32), we can see that the (IOS) gain with respect to the inputs $\dot{\hat{\mathbf{q}}}_i^s$ and $\Delta\hat{\mathbf{q}}_i$ are, respectively, $\gamma_i^{\{1\}} = \lambda_{i_{\max}}/(\lambda_{i_{\min}}\cdot\kappa_i\cdot\mu)$ and $\gamma_i^{\{2\}} = (\lambda_{i_{\max}}\cdot\alpha_0)/(\lambda_{i_{\min}}\cdot\kappa_i\cdot\mu)$.

At this point, it is worth mentioning that in the case where one (or some) system(s) in the team, say for example the l^{th} system, does not receive information from other neighbours, *i.e.,* $\kappa_l = 0$, we can easily verify from the dynamics of the filter (16) that $\psi_l^{\{1\}} \to \psi_l^{\{2\}} \to 0$ exponentially. This leads us to the conclusion that $\dot{\mathbf{q}}_l \to 0$ and $\tilde{\mathbf{q}}_l \to 0$ in view of (15), (17), and the fact that $\mathbf{s}_l \to 0$. Therefore, this l^{th} system, with $\kappa_l = 0$, can be considered to be similar to (30), and is ISS with the corresponding IOS gains with respect to the inputs $\dot{\hat{\mathbf{q}}}_i^s$ and $\Delta\hat{\mathbf{q}}_i$ are equal to zero.

Now, using Assumption 1, the following estimates on $\left|\dot{\hat{\mathbf{q}}}_i^s\right|$ and $|\Delta\hat{\mathbf{q}}_i|$ can be derived

$$\begin{aligned} \left|\dot{\hat{\mathbf{q}}}_i^s(t)\right| &= \left|\sum_{j=1}^{n} k_{ij}\dot{\mathbf{q}}_j\left(t - \tau_{ij}^s(t)\right)\left[1 - \frac{d\tau_{ij}^s(t)}{dt}\right]\right| \\ &\leq \sum_{j=1}^{n} k_{ij}\left(1 + \Upsilon_{ij}\right)\left|\dot{\mathbf{q}}_j\left(t - \tau_{ij}^s(t)\right)\right|, \end{aligned} \tag{33}$$

$$\begin{aligned} |\Delta\hat{\mathbf{q}}_i| &= \left|\sum_{j=1}^{n} k_{ij}\left[\mathbf{q}_j(t - \tau_{ij}(t)) - \mathbf{q}_j\left(t - \tau_{ij}^s(t)\right)\right]\right| \\ &\leq \sum_{j=1}^{n} k_{ij}\left|\mathbf{q}_j(t - \tau_{ij}(t)) - \mathbf{q}_j\left(t - \tau_{ij}^s(t)\right)\right| \\ &\leq \sum_{j=1}^{n} k_{ij}\cdot\Delta_{ij}^{\tau}\cdot\left(\sup_{\sigma\in[t_1,t_2]}|\dot{\mathbf{q}}_j(\sigma)|\right), \end{aligned} \tag{34}$$

with $t_1 = (t - \max\{\tau_{ij}(t), \tau_{ij}^s(t)\})$ and $t_2 = (t - \min\{\tau_{ij}(t), \tau_{ij}^s(t)\})$.

It can be seen from inequalities (33) and (34) that the interconnected subsystems (30) satisfy condition (12) in Theorem 1 with respect to the inputs $\dot{\hat{\mathbf{q}}}_i^s$ and $\Delta\hat{\mathbf{q}}_i$. In addition, the interconnection gains with respect to the inputs $\dot{\hat{\mathbf{q}}}_i^s$ and

$\Delta\hat{\mathbf{q}}_i$ can be obtained, respectively, from (33) and (34), as: $\mu_{ij}^{\{1\}} = k_{ij}(1+\Upsilon_{ij})$ and $\mu_{ij}^{\{2\}} = k_{ij}\Delta_{ij}^{\tau}$.

As a result, we can write the overall gain from $\dot{\mathbf{q}}_j$ to $\dot{\mathbf{q}}_i$ as

$$\gamma_{ij} := \frac{\lambda_{i_{\max}}}{\lambda_{i_{\min}} \cdot \kappa_i \cdot \mu} \cdot k_{ij}\left(1+\Upsilon_{ij}+\alpha_0 \cdot \Delta_{ij}^{\tau}\right). \tag{35}$$

As noted above, if there are some systems that do not receive information from other neighbours, then the corresponding gain is simply null, *i.e.,* $\gamma_{lj} = 0$, for each $l \in \mathcal{N}$ with $\kappa_l = 0$.

Then, using the result in Theorem 1, we conclude that the trajectories of the interconnected system are bounded and converge to zero if

$$\rho\left(\Gamma\right) < 1, \tag{36}$$

where $\Gamma := \{\gamma_{ij}\}_{i,j\in\mathcal{N}}$, γ_{ij} is given in (35), and $\rho(\Gamma)$ is the spectral radius of matrix Γ. Since $k_{ii} = 0$ for all $i \in \mathcal{N}$, Geršgorin disc theorem [19] implies that (19) is a sufficient condition for (36). Thus, we conclude that $\tilde{\psi}_i$, $\psi_i^{\{2\}}$, $\tilde{\mathbf{q}}_i$ are bounded and $\tilde{\psi}_i \to 0$, $\psi_i^{\{2\}} \to 0$, $\tilde{\mathbf{q}}_i \to 0$, for $i \in \mathcal{N}$. Also, since $\mathbf{s}_i \to 0$, we know from (31) that $\dot{\mathbf{q}}_i \to 0$ for $i \in \mathcal{N}$.

Furthermore, $\dot{\mathbf{q}}_i \to 0$ implies from (34) that $\Delta\hat{\mathbf{q}}_i \to 0$. This with the fact that $\tilde{\psi}_i \to 0$ lead us to conclude from (24) and (25) that $(\psi_i^{\{1\}} - \hat{\mathbf{q}}_i^s) \to (\psi_i^{\{1\}} - \hat{\mathbf{q}}_i) \to 0$ for $i \in \mathcal{N}$. Also, since we have shown that $\tilde{\mathbf{q}}_i \to 0$, we conclude from (15) that $(\kappa_i\mathbf{q}_i - \hat{\mathbf{q}}_i) \to 0$. Consequently, we know from (22) that $\sum_{j=1}^{n} k_{ij}\left(\mathbf{q}_i - \mathbf{q}_j(t-\tau_{ij}(t))\right) \to 0$, for $i \in \mathcal{N}$, and equivalently

$$\sum_{j=1}^{n} k_{ij}\left(\mathbf{q}_i - \mathbf{q}_j - \int_{t-\tau_{ij}(t)}^{t} \dot{\mathbf{q}}_j(s)ds\right) \to 0, \tag{37}$$

for $i \in \mathcal{N}$, where we have used: $(\mathbf{q}_j - \mathbf{q}_j(t-\tau_{ij}(t))) = \int_{t-\tau_{ij}(t)}^{t} \dot{\mathbf{q}}_j(s)ds$.

Combining (2), (6), (7), and taking into account the assumption $\limsup\limits_{t\to+\infty} \tau^*(t) < \infty$ and the fact that $\dot{\mathbf{q}}_i \to 0$, we see that the integral term in the left-hand side of (37) converges asymptotically to zero. As a result, we conclude that $\sum_{j=1}^{n} k_{ij}(\mathbf{q}_i - \mathbf{q}_j) \to 0$, for $i \in \mathcal{N}$, which is equivalent to $(\mathcal{L} \otimes \mathbf{I}_m)\mathbf{Q} \to 0$ where $\mathcal{L}$ is the Laplacian matrix of the communication graph $\mathcal{G}$, $\mathbf{Q} \in \mathbb{R}^{nm}$ is the vector containing all $\mathbf{q}_i$ for $i \in \mathcal{N}$, and $\otimes$ is the Kronecker product. Therefore, following arguments similar to the ones presented in [15], under the condition that the communication graph contains a spanning tree, we can conclude that $(\mathbf{q}_i - \mathbf{q}_j) \to 0$ for all $i, j \in \mathcal{N}$. The proof is complete. ∎

V. Simulation results

In this section, simulation results of a network of ten planar two degrees of freedom rigid manipulator arms with revolute joints governed by similar dynamics are presented. Each manipulator is described by the Euler-Lagrange equations of the form (1), where $\mathbf{q} = (\theta_1, \theta_2)^T \in \mathbb{R}^2$. The inertia matrix $\mathbf{M}(\mathbf{q}) = [m_{jk}]_{2\times 2}$ consists of the following elements: $m_{11} = m_1 l^2 c + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(\theta_2)) + I_1 + I_2$, $m_{12} = m_{21} = m_2(l_{c2}^2 + l_1 l_{c2}\cos(\theta_2)) + I_2$, and $m_{22} = m_2 l_{c2}^2 + I_2$. The matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = [c_{jk}]_{2\times 2}$ is given by $c_{11} = h\dot{\theta}_2$, $c_{12} = h(\dot{\theta}_1 + \dot{\theta}_2)$, $c_{21} = -h\dot{\theta}_1$, and $c_{22} = 0$, with $h = -m_2 l_1 l_{c2}\sin(\theta_2)$. The gravitational force vector $G(\mathbf{q}) = [G_1, G_2]^T$ is given by $G_1 = (m_1 l_{c1} + m_2 l_1)g\cos(\theta_1) + m_2 l_{c2} g\cos(\theta_1 + \theta_2)$ and $G_2 = m_2 l_{c_2} g\cos(\theta_1 + \theta_2)$. The model parameters are $m_1 = m_2 = 1$ kg, $l_1 = l_2 = 0.5$ m, $l_{c1} = l_{c2} = 0.25$ m, $I_1 = I_2 = 0.1$ kg/m^2, and $g = 9.81$ m/sec^2. The communication topology between agents is represented by the directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{K})$, with $\mathcal{N} = \{1, \ldots, 10\}$, $\mathcal{E} = \{(1,2),(1,10),(2,3),(4,1),(4,5),(5,6),(6,7),(7,8),(8,7),(8,9),(9,1),(9,3),(9,10),(10,8)\}$ and the adjacency matrix $\mathcal{K} = [k_{ij}]$, with $k_{ij} = 2$, for $(j,i) \in \mathcal{E}$, and zero otherwise. It can be easily verified that $\mathcal{G}$ contains a spanning tree.

We implement the controller (14)-(16), where the filter (16) is initialized as: $\psi_i^{\{1\}}(0) = (1,1)^\top$, $\psi_i^{\{2\}}(0) = (2,1)^\top$, for $i \in \mathcal{N}$, and the gains are $k_i^s = 2$, $\Lambda_i = 1.5\mathbf{I}_2$, for $i \in \mathcal{N}$, $\alpha_0 = 2.25$ and $\alpha_1 = 3$. This choice of the gains guarantees that $\mu = 1.5$ as defined by (18).

We consider the following communication delays between each pair of communicating systems: $\tau_{ij}(t) = \bar{\tau}_{ij}\phi(t)$ sec, with the constants $\bar{\tau}_{1i} = 0.6$, $\bar{\tau}_{2i} = 0.6$, $\bar{\tau}_{3i} = 0.6$, $\bar{\tau}_{4i} = 0.6$, $\bar{\tau}_{5i} = 0.5$, $\bar{\tau}_{6i} = 0.5$, $\bar{\tau}_{7i} = 0.6$, $\bar{\tau}_{8i} = 0.5$, $\bar{\tau}_{9i} = 0.6$, $\bar{\tau}_{10i} = 0.6$, for $i \in \mathcal{N}$, and the function $\phi(t) = (1 - \cos(0.25\ t) + 0.25\ r(t))$, where $r(t)$ is a uniform random function; $r(t) \in [0,1]$. It is clear that $\tau_{ij}(t)$ given above satisfies Assumption 1. In addition, we can verify by simple computations that condition (19) is satisfied with $\Upsilon_{ij} = \Delta_{ij}^{\tau} = 0.15$. The obtained results in this case are illustrated in Fig. 1-3, where we can see that all systems synchronize in the presence of time-varying communication delays. Note that the considered communication delays are non-differentiable and bounded, however, their upper bounds are not explicitly required to satisfy the obtained conditions in this work.

VI. Conclusions

A multi-dimensional IOS small gain approach has been used to solve the synchronization problem of Euler-Lagrange systems in the presence of irregular communication delays. As a result, conditions are obtained that, in particular, allow to guarantee the synchronization in the presence of discontinuous possibly unbounded communication delays under the assumption that the directed communication graph contains a spanning tree. The simulation results confirm the theoretical developments. Possible directions for future research include the extension of these results to the case of parametric uncertainties, more irregular communication delays and significant information losses, as well as synchronization with prescribed nonzero velocities.

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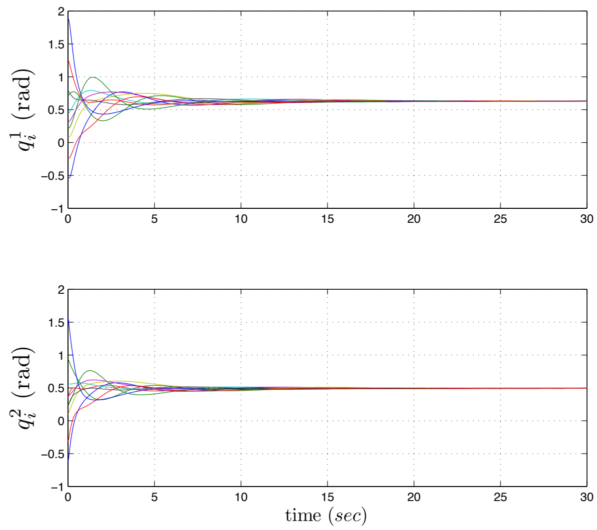

Fig. 1. Joint angles with $\mathbf{q}_i = (q_i^1, q_i^2)^T$ and $i \in \mathcal{N}$.

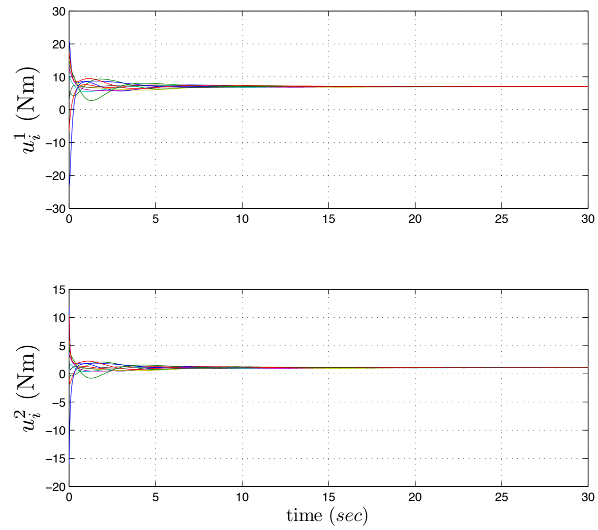

Fig. 3. Input torques with $\mathbf{\Gamma}_i = (\tau_i^1, \tau_i^2)^T$ and $i \in \mathcal{N}$.

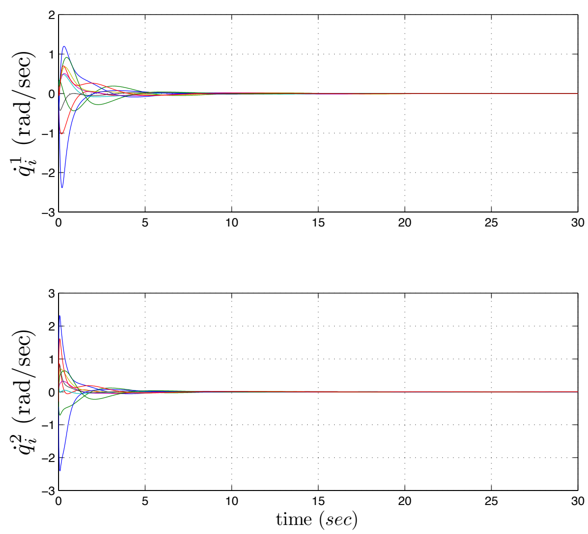

Fig. 2. Joint angles derivatives with $\dot{\mathbf{q}}_i = (\dot{q}_i^1, \dot{q}_i^2)^T$ and $i \in \mathcal{N}$.

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