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Iterative learning control for non-linear systems described by a blended multiple model representation

ABDELHAMID TAYEBI†* and MAREK B. ZAREMBA‡

This paper deals with the design of gain-scheduling-based iterative learning controllers for continuous-time non-linear systems described by a blended multiple model representation. Sufficient conditions guaranteeing the convergence of the infinity norm as well as the λ-norm of the tracking error are derived. The effectiveness of the proposed control scheme is illustrated on an example of a non-affine-in-input system.

1. Introduction

Developing a systematic control design procedure for non-linear systems is generally a hard task. In practice, the control of such systems is often achieved through the use of a linear controller based upon the linearization of the underlying non-linear system about an operating point. However, the local controller will be valid only in the neighbourhood of this operating point. In order to describe the non-linear system behaviour in a wide operating range, one can use a number of local models over different operating points. The combination of all these local models by means of an adequate smooth interpolation method leading to a global approximating model for the non-linear system is referred to as a blended multiple model representation (BMMR). It is also called a local model network (Johansen *et al.* 1993) or a fuzzy model (Tanaka *et al.* 1998) in the literature. This procedure can be viewed as a generalization of the standard basis function networks where, in essence, the local models are constant output values. The approximation properties of the BMMR have been examined by several authors in the literature, and it has been demonstrated that, under smoothness conditions on the non-linear function, this structure can uniformly approximate a non-linear model (Wang *et al.* 1992, Johansen *et al.* 1993, Shorten *et al.* 1999). The idea of multiple model representation, which has been known for several years (Poggio *et al.* 1990), was extended for modelling and control purposes (Johansen *et al.* 1993, Gawthrop *et al.* 1995, Gollee and Hunt 1997, Hunt and Johansen 1997, Prasad *et al.* 1998, Tanaka *et al.* 1998, Kiriakidis 1999, Sluphaug *et al.* 1999).

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* Author for correspondence. e-mail: Abdelhamid.Tayebi@lakeheadu.ca

† Department of Electrical Engineering, Lakehead University, Thunder Bay, Ontario P7B 5E1, Canada.

‡ Département d'informatique, Université du Québec, Hull, Québec J8X 3X7, Canada.

This paper addresses the issue of designing iterative learning control (ILC) for a class of non-linear systems described by a blended multiple model representation. In its principle, the iterative learning control technique (Arimoto *et al.* 1984, Sugie and Ono 1991, Kuc *et al.* 1992, Ahn *et al.* 1993, Kurek and Zaremba 1993, Jang *et al.* 1995, Moore 1998, Xu and Viswanathan 2000) aims to improve the transient response and the tracking performance of systems that execute the same trajectory or operation over and over again. It consists in finding an adequate rule which allows the controller to learn from the tracking errors of the previous operations and perform progressively better with every new operation in order to achieve accurate tracking when the number of iterations increase.

In this paper, we propose two gain-scheduling-based ILC schemes for non-linear systems described by a BMMR. The first scheme is a P-type ILC designed for systems with direct input–output transmission, while the second one is a D-type ILC designed for systems without direct input–output transmission. Sufficient conditions guaranteeing the convergence of the infinity norm as well as the λ-norm of the tracking error are derived.

The proposed approach allows for efficient control of non-linear systems including the class of non-affine-in-input systems—which is well known as a challenging class from the control point of view. To the best of our knowledge, there is no ILC scheme in the literature dealing with this particular class of non-linear systems. In this paper, we address this problem through the control of a class of non-linear systems described by a BMMR—which can be viewed as a quasi-global approximation of non-linear systems including the class of non-affine-in-input systems. ILC design for a system belonging to that particular class is illustrated in a numerical example in the final part of the paper.

2. Problem statement

Let us consider a non-linear system operating repeatedly over the time interval $[0, T]$, and described by the following BMMR

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$$\dot{x}_k(t) = \sum_{i=1}^{N} (f(x_i^e, u_i^e) + A_i(x_k(t) - x_i^e) + B_i(u_k(t) - u_i^e))\rho_i(z_k(t))$$
$$y_k(t) = \sum_{i=1}^{N} (y_i^e + C_i(x_k(t) - x_i^e) + D_i(u_k(t) - u_i^e))\rho_i(z_k(t)) \quad (1)$$

Where k is the iteration index or the operation number, and $t \in [0, T]$ is time. Vector $x_k(t) \in X \subset \mathbb{R}^n$ is the state vector, $u_k(t) \in U \subset \mathbb{R}^m$ is the control vector, $y_k(t) \in \mathbb{R}^p$ is the system output and $z_k(t) = H(x_k(t), u_k(t))$ represents the actual operating point.

Parameters (x_i^e, u_i^e), $i = 1, \ldots, N$, which might be time-varying, are generally the operating points about which the original non-linear system has been linearized, y_i^e is the output at the operating point i, and $f : X \times U \to \mathbb{R}^n$ is a given non-linear function. Matrices A_i, B_i, C_i and D_i are generally obtained from a non-linear system using a first-order Taylor expansion about (x_i^e, u_i^e).

Functions ρ_i, generally called validity functions or fuzzy basis functions, are used to interpolate the N local models leading to a quasi-global approximation in the form of (1) of a given non-linear system. The validity functions are positive semi-definite with the following properties

(i) $\sum_{i=1}^{N} \rho_i(z_k) = 1$
(ii) $\rho_i(z_k) \to 1$ as $z_k \to z_i^e$ (where the local model i is an accurate description of the system), and $\rho_i(z_k) \to 0$ elsewhere.

The function $\rho_i(z_k)$ is usually chosen as a normalized Gaussian function describing the distance between the current operating point $z_k(t)$ and the local operating point z_i^e

$$\rho_i(z_k(t)) = \frac{\exp\left[-0.5(z_k(t) - z_i^e)^{\mathrm{T}} \Sigma_i (z_k(t) - z_i^e)\right]}{\sum_{i=1}^{N} \exp[-0.5(z_k(t) - z_i^e)^{\mathrm{T}} \Sigma_i (z_k(t) - z_i^e)]} \quad (2)$$

where Σ_i is a positive definite scaling matrix. Obviously, the overlapping of operating regimes allows the validity functions to assure a smooth transfer from one region of the model to the next.

Our objective is to derive an iterative control law $u_k(t)$, starting from any arbitrarily continuous and bounded input $u_0(t)$ over $[0, T]$, such that the output $y_k(t)$ converges to the desired output $y_d(t)$, for all $t \in [0, T]$, when the iteration index k tends to infinity.

Throughout the paper, we will use the norm defined as $\|V\| = \max_{1 \le i \le n} |v_i|$ for a vector $V = [v_1, \ldots, v_n]^{\mathrm{T}}$, and as

$$\|M\| = \max_{1 \le i \le m} \sum_{j=1}^{n} |m_{i,j}|$$

for a matrix $M = [m_{i,j}] \in \mathbb{R}^{m \times n}$. We will also use the infinity norm defined as $\|\star(t)\|_\infty = \sup_{t \in [0,T]} \|\star(t)\|$, and the λ-norm defined as $\|\star(t)\|_\lambda = \sup_{t \in [0,T]} \mathrm{e}^{-\lambda t} \|\star(t)\|$. Note that $\|\star\|_\lambda \le \|\star\|_\infty \le \mathrm{e}^{\lambda T} \|\star\|_\lambda$ for any $\lambda > 0$.

3. Iterative learning control synthesis

In the balance of this paper, we will make use of the following assumptions

A1. There exists a control law $u_d(t)$ leading to the desired state $x_d(t)$ and to the desired output $y_d(t)$. The signals $u_d(t)$ and $x_d(t)$ are bounded on $[0, T]$.
A2. The initial error is equal to zero, i.e. $x_k(0) = x_d(0)$, for all k.
A3. For all $(x_k, u_k) \in X \times U$ and $t \in [0, T]$, there exists a constant K_i satisfying

$$\|\rho_i(z_d) - \rho_i(z_k)\| \le K_i \|x_d - x_k\|, \qquad i = 1, \ldots, N$$

where $z_k = H(x_k(t), u_k(t))$ and $z_d = H(x_d(t), u_d(t))$.

Assumptions A1 and A2 are well known in the literature concerning iterative learning control, whereas Assumption A3 is related to the validity functions involved in the BMMR. This last assumption is not very restrictive, since if we consider, for example, the validity function given in equation (2), one can easily check that its partial derivative with respect to z_k is uniformly continuous, and it is bounded on the bounded set

$$D_{r,i} = \{z_k \in \mathbb{R}^n / \|z_k - z_i\| \le r\} \qquad \forall r > 0$$

that is

$$\left\| \frac{\partial \rho_i(z_k)}{\partial z_k} \right\| \le 2\|\Sigma_i\| \|z_k - z_i\|$$

which means that $\rho_i(z_k)$ is locally Lipschitz on D_i for any $r > 0$. Hence, Assumption A3 is fulfilled if $H(x_k, u_k)$ is uniformly Lipschitz with respect to x_k. In conclusion, if $\rho_i(z_k)$ is taken as in (2), Assumption A3 will be fulfilled if $H(x_k, u_k)$ is uniformly Lipschitz with respect to x_k. This is not a restrictive condition, because $H(x_k, u_k)$ is chosen by the designer to determine the operating point.

It is worth noting that Assumptions A1–A3 will be needed to prove the convergence to zero of the tracking error (Theorems 1 and 3), whereas Assumption A1 alone will be needed to prove the convergence of the tracking error to a residual domain around zero (Theorem 2).

Let us consider the iterative controller

$$u_{k+1}(t) = u_k(t) + \left(\sum_{i=1}^{N} G_i\rho_i(z_k)\right)e_k(t) \qquad (3)$$

where $e_k(t) = y_d(t) - y_k(t)$ represents the tracking error and G_i, $i = 1, \ldots, N$ are the control gains to be designed to ensure the convergence of the iterative process when k tends to infinity. This scheme has a strong link with gain scheduling control. In fact, in gain scheduling techniques, the control gains change as a function of an auxiliary scheduling variable (Shamma and Athans 1990, Rugh, 1991, Leith and Leithead 2000) while in the ILC scheme (3), the control gain $G_k = \sum_{i=1}^{N} G_i\rho_i(z_k)$ changes smoothly according to the values of the validity functions depending on the operating point.

Before stating Theorem 1, let us define the parameters

$$b = \left\|\sum_{i=1}^{N} B_i\right\|, \qquad g = \left\|\sum_{i=1}^{N} G_i\right\|, \qquad c = \left\|\sum_{i=1}^{N} C_i\right\|$$

$$a = \sup_{t\in[0,T]}\left\{\sum_{i=1}^{N} K_i\|f(x_i^e, u_i^e) + A_i(x_d(t) - x_i^e) + B_i(u_d(t) - u_i^e)\|\right\} + \left\|\sum_{i=1}^{N} A_i\right\|$$

$$\psi = gb\left(c + \sup_{t\in[0,T]}\left\{\sum_{i=1}^{N} K_i\|y_i^e + C_i(x_d(t) - x_i^e) + D_i(u_d(t) - u_i^e)\|\right\}\right)$$

$$\alpha = \left\|I - \left(\sum_{i=1}^{N} G_i\rho_i(z_k)\right)\left(\sum_{i=1}^{N} D_i\rho_i(z_k)\right)\right\| \quad \text{and}$$

$$\beta = \frac{\psi}{a}(\mathrm{e}^{aT} - 1)$$

Theorem 1: *Consider system* (1) *with the ILC* (3). *If Assumptions A1–A3 are fulfilled, then, for all* $t \in [0, T]$, *the following hold:*

(i) *If* $\sup_{t\in[0,T]}\{\alpha\} < 1$, *there exists* $\lambda > a + \psi$ *such that* $\|y_d(t) - y_k(t)\|_\infty$ *tends to zero when* k *tends to infinity, with a rate of convergence less than or equal to*

$$\gamma_1 = \mathrm{e}^{\lambda T}\left(\sup_{t\in[0,T]}\{\alpha(t)\} + \frac{\psi}{\lambda - a}(1 - \mathrm{e}^{(a-\lambda)T})\right)^k$$

(ii) *If* $\sup_{t\in[0,T]}\{\alpha(t)\} + \beta < 1$, *then* $\|y_d(t) - y_k(t)\|_\infty$ *converges to zero when* k *tends to infinity, with a rate of convergence less than or equal to*

$$\gamma_2 = \left(\sup_{t\in[0,T]}\{\alpha(t)\} + \frac{\psi}{a}(\mathrm{e}^{aT} - 1)\right)^k$$

Proof: Let $\tilde{u}_k(t)$ denote the control error at the kth iteration. Hence, the control error at the $(k+1)$th iteration can be expressed as

$$\tilde{u}_{k+1}(t) = u_d(t) - u_{k+1}(t) = u_d(t) - u_k(t) - \left(\sum_{i=1}^{N} G_i\rho_i(z_k)\right)e_k(t) \qquad (4)$$

which gives

$$\tilde{u}_{k+1}(t) = \tilde{u}_k(t) - \left(\sum_{i=1}^{N} G_i\rho_i(z_k)\right)e_k(t). \qquad (5)$$

The tracking error can be expressed as

$$\begin{aligned} e_k(t) &= \sum_{i=1}^{N}(y_i^e + C_i(x_d - x_i^e) + D_i(u_d - u_i^e))\rho_i(z_d) \\ &\quad - \sum_{i=1}^{N}(y_i^e + C_i(x_k - x_i^e) + D_i(u_k - u_i^e))\rho_i(z_k) \\ &= \sum_{i=1}^{N}(C_i\tilde{x}_k + D_i\tilde{u}_k)\rho_i(z_k) + \sum_{i=1}^{N}(y_i^e + C_i(x_d - x_i^e) \\ &\quad + D_i(u_d - u_i^e))(\rho_i(z_d) - \rho_i(z_k)) \end{aligned} \qquad (6)$$

where $\tilde{x}_k = x_d - x_k$. Hence, equation (5) becomes

$$\begin{aligned} \tilde{u}_{k+1}(t) &= \left(I - \sum_{i=1}^{N} G_i\rho_i(z_k)\sum_{i=1}^{N} D_i\rho_i(z_k)\right)\tilde{u}_k(t) \\ &\quad - \left(\sum_{i=1}^{N} G_i\rho_i(z_k)\sum_{i=1}^{N} C_i\rho_i(z_k)\right)\tilde{x}_k(t) \\ &\quad - \sum_{i=1}^{N} G_i\rho_i(z_k)\sum_{i=1}^{N}(y_i^e + C_i(x_d - x_i^e) \\ &\quad + D_i(u_d - u_i^e))(\rho_i(z_d) - \rho_i(z_k)) \end{aligned} \qquad (7)$$

Since $\sum_{i=1}^{N}\rho_i(z_k) = 1$, from the latter one can obtain the inequality

$$\begin{aligned} \|\tilde{u}_{k+1}(t)\| &\le \alpha\|\tilde{u}_k(t)\| + \left\|\sum_{i=1}^{N} G_i\right\|\left\|\sum_{i=1}^{N} C_i\right\|\|\tilde{x}_k(t)\| \\ &\quad + \left\|\sum_{i=1}^{N} G_i\right\|\sum_{i=1}^{N}\|y_i^e + C_i(x_d - x_i^e) \\ &\quad + D_i(u_d - u_i^e)\|\|\rho_i(z_d) - \rho_i(z_k)\| \end{aligned} \qquad (8)$$

Now, using Assumption A3, one has

$$\|\tilde{u}_{k+1}(t)\| \leq \alpha\|\tilde{u}_k(t)\| + \left\|\sum_{i=1}^{N} G_i\right\|\left(\left\|\sum_{i=1}^{N} C_i\right\| + \sum_{i=1}^{N} K_i\|y_i^e + C_i(x_d - x_i^e) + D_i(u_d - u_i^e)\|\right)\|\tilde{x}_k(t)\| \tag{9}$$

In view of Assumption A2 and system (1), the state error is given by

$$\tilde{x}_k(t) = \int_0^t (\dot{x}_d(\tau) - \dot{x}_k(\tau))\mathrm{d}\tau \tag{10}$$

which leads to

$$\begin{aligned}\tilde{x}_k(t) &= \int_0^t \sum_{i=1}^{N}(f(x_i^e, u_i^e) + A_i(x_d - x_i^e) + B_i(u_d - u_i^e))\rho_i(z_d)\mathrm{d}\tau \\ &\quad - \int_0^t \sum_{i=1}^{N}(f(x_i^e, u_i^e) + A_i(x_k - x_i^e) + B_i(u_k - u_i^e))\rho_i(z_d)\mathrm{d}\tau \qquad (11)\\ &= \int_0^t \sum_{i=1}^{N}(A_i\tilde{x}_k(\tau) + B_i\tilde{u}_k(t))\rho_i(z_k)\mathrm{d}\tau \\ &\quad - \int_0^t \sum_{i=1}^{N}(f(x_i^e, u_i^e) + A_i(x_d - x_i^e) + B_i(u_d - u_i^e))(\rho_i(z_d) - \rho_i(z_k))d\tau.\end{aligned}$$

In view of Assumption A1, i.e. x_d and u_d are bounded, and Assumption A3, one has

$$\|\tilde{x}_k(t)\| \leq \int_0^t (a\|\tilde{x}_k(\tau)\| + b\|\tilde{u}_k(\tau)\|)\mathrm{d}\tau \tag{12}$$

Applying the Bellman–Gronwall Lemma we get

$$\|\tilde{x}_k(t)\| \leq \int_0^t b\|\tilde{u}_k(\tau)\|\mathrm{e}^{a(t-\tau)}\mathrm{d}\tau \tag{13}$$

Then, in view of (13), inequality (9) becomes

$$\|\tilde{u}_{k+1}(t)\| \leq \alpha\|\tilde{u}_k(t)\| + \psi\int_0^t \|\tilde{u}_k(\tau)\|\mathrm{e}^{a(t-\tau)}\mathrm{d}\tau \tag{14}$$

which leads to

$$\sup_{t\in[0,T]}\|\tilde{u}_{k+1}(t)\| \leq \sup_{t\in[0,T]}\{\alpha\}\sup_{t\in[0,T]}\|\tilde{u}_k(t)\| + \sup_{t\in[0,T]}\left\{\psi\int_0^t \|\tilde{u}_k(\tau)\|\mathrm{e}^{a(t-\tau)}d\tau\right\} \tag{15}$$

and

$$\sup_{t\in[0,T]}\|\tilde{u}_{k+1}(t)\| \leq \left(\sup_{t\in[0,T]}\{\alpha\} + \psi\sup_{t\in[0,T]}\int_0^t \mathrm{e}^{a(t-\tau)}d\tau\right) \times \sup_{t\in[0,T]}\|\tilde{u}_k(t)\| \tag{16}$$

i.e.

$$\sup_{t\in[0,T]}\|\tilde{u}_{k+1}(t)\| \leq \left(\sup_{t\in[0,T]}\{\alpha\} + \beta\right)\sup_{t\in[0,T]}\|\tilde{u}_k(t)\| \tag{17}$$

Thus,

$$\sup_{t\in[0,T]}\|\tilde{u}_k(t)\| \leq \left(\sup_{t\in[0,T]}\{\alpha\} + \beta\right)^k \sup_{t\in[0,T]}\|\tilde{u}_0(t)\| \tag{18}$$

Finally, if $\sup_{t\in[0,T]}\{\alpha\} + \beta < 1$, it is obvious that $\|\tilde{u}_k\|_\infty \to 0$ as k tends to infinity, which in terms of Assumption A1 implies that $\|y_d(t) - y_k(t)\|_\infty \to 0$ as k tends to infinity.

Now, let us investigate the case where the condition $0 \leq \beta < 1$ is not fulfilled. We will show that even if this condition is not satisfied, one can guarantee the convergence of the λ-norm of the tracking error.

Multiplying each side of (14) by $\mathrm{e}^{-\lambda t}$, $\lambda > a$, and applying the λ-norm gives

$$\|\tilde{u}_{k+1}(t)\|_\lambda \leq \sup_{t\in[0,T]}\{\alpha\}\|\tilde{u}_k(t)\|_\lambda + \sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\psi\int_0^t \|\tilde{u}_k(\tau)\|\mathrm{e}^{a(t-\tau)}\mathrm{d}\tau\right\} \tag{19}$$

$$\|\tilde{u}_{k+1}(t)\|_\lambda \leq \sup_{t\in[0,T]}\{\alpha\}\|\tilde{u}_k(t)\|_\lambda + \sup_{t\in[0,T]}\left\{\psi\int_0^t \mathrm{e}^{(a-\lambda)(t-\tau)}\mathrm{d}\tau\right\}\|\tilde{u}_k(t)\|_\lambda \tag{20}$$

which leads to

$$\|\tilde{u}_{k+1}(t)\|_\lambda \leq \left(\sup_{t\in[0,T]}\{\alpha\} + \frac{\psi}{\lambda - a}(1 - \mathrm{e}^{(a-\lambda)T})\right)\|\tilde{u}_k(t)\|_\lambda \tag{21}$$

that is

$$\|\tilde{u}_k(t)\|_\lambda \le \left(\sup_{t\in[0,T]}\{\alpha\} + \frac{\psi}{\lambda - a}(1 - \mathrm{e}^{(a-\lambda)T})\right)^k \|\tilde{u}_0(t)\|_\lambda \tag{22}$$

If $\sup_{t\in[0,T]}\{\alpha\} < 1$, then there exists $\lambda > a + \psi$ such that

$$\left(\sup_{t\in[0,T]}\{\alpha\} + \frac{\psi}{\lambda - a}(1 - \mathrm{e}^{(a-\lambda)T})\right) < 1$$

This implies that $\|\tilde{u}_k\|_\lambda \to 0$ as k tends to infinity, which, according to Assumption A1, ensures that $\|y_d(t) - y_k(t)\|_\lambda \to 0$ as k tends to infinity for all $t \in [0, T]$. Considering the definition of the λ-norm, one can conclude that $\|\tilde{u}_k(t)\|_\infty \le \gamma_1 \|\tilde{u}_0(t)\|_\infty$, where

$$\gamma_1 = \mathrm{e}^{\lambda T}\left(\sup_{t\in[0,T]}\{\alpha(t)\} + \frac{\psi}{\lambda - a}(1 - \mathrm{e}^{(a-\lambda)T})\right)^k$$

which implies that $\|\tilde{u}_k(t)\|_\infty$ and $\|y_d(t) - y_k(t)\|_\infty$ converge to zero when k tends to infinity. □

Now consider the case where Assumptions A2 and A3 are not fulfilled. In this case, one can always ensure the convergence of the tracking error to a certain residual domain around zero depending on the system parameters and the control gains G_i.

Assume that $\|x_k(0) - x_d(0)\| \le \xi_2$ for all k. Since the validity functions $\rho_i(.)$ belong to $[0, 1]$, it is clear that $\|\rho_i(z_d) - \rho_i(z_k)\| \le \xi_1$, $0 \le \xi_1 \le 1$, for all $k \in \mathbb{N}$ and $t \in [0, T]$. Before stating Theorem 2, let us define the parameters

$$m_1 = \left\|\sum_{i=1}^N G_i\right\| \left\|\sum_{i=1}^N C_i\right\|, \qquad m_2 = \left\|\sum_{i=1}^N A_i\right\|,$$

$$m_3 = \left\|\sum_{i=1}^N B_i\right\|, \qquad m_4 = \frac{m_1 m_3}{m_2}(\mathrm{e}^{m_2 T} - 1)$$

$$m_5 = \frac{m_1 \epsilon_2}{m_2}(\mathrm{e}^{m_2 T} - 1) + m_1 \xi_2 \mathrm{e}^{m_2 T} + \epsilon_1$$

$$m_6 = \epsilon_1 + m_1 \xi_2 + \frac{m_1 \epsilon_2}{m_2} \sup_{t\in[0,T]}\{\mathrm{e}^{(m_2-\lambda)t} - \mathrm{e}^{-\lambda t}\} \le \epsilon_1 + m_1 \xi_2 + \frac{m_1 \epsilon_2}{m_2}$$

$$\epsilon_1 = \xi_1 \sup_{t\in[0,T]}\left\{\left\|\sum_{i=1}^N G_i\right\| \sum_{i=1}^N \|y_i^e + C_i(x_d(t) - x_i^e) + D_i(u_d(t) - u_i^e)\|\right\}$$

$$\epsilon_2 = \xi_1 \sup_{t\in[0,T]}\left\{\sum_{i=1}^N \|f(x_i^e, u_i^e) + A_i(x_d(t) - x_i^e) + B_i(u_d(t) - u_i^e)\|\right\}$$

$$l_1 = \sup_{t\in[0,T]}\{\alpha\} + m_4$$

$$l_2 = \sup_{t\in[0,T]}\{\alpha\} + \frac{m_1 m_3}{\lambda - m_2}(1 - \mathrm{e}^{(m_2-\lambda)T})$$

Theorem 2: *Consider system* (1) *with the ILC* (3). *If Assumption A1 is fulfilled and* $\|x_k(0) - x_d(0)\| \le \xi_2$ *for all* k, *then the following hold for all* $t \in [0, T]$.

(i) *If* $\sup_{t\in[0,T]}\{\alpha\} < 1$, *there exists* $\lambda > m_2 + m_1 m_3$ *such that*

$$\lim_{k\to\infty} \|\tilde{u}_k\|_\infty \le \mathrm{e}^{\lambda T}\frac{m_6}{1 - l_2}$$

and

$$\lim_{k\to\infty}\|\tilde{e}_k\|_\infty \le \left\|\sum_{i=1}^N C_i\right\| \left(\xi_2 \mathrm{e}^{m_2 T} + \left(\frac{m_3 m_6 \mathrm{e}^{\lambda T}}{m_2(1 - l_2)} + \frac{\epsilon_2}{m_2}\right) \times (\mathrm{e}^{m_2 T} - 1)\right) + \left\|\sum_{i=1}^N D_i\right\| \left(\frac{m_6}{1 - l_2}\right)\mathrm{e}^{\lambda T} + \xi_1 \left\|\sum_{i=1}^N (y_i^e + C_i(x_d(t) - x_i^e) + D_i(u_d(t) - u_i^e))\right\|_\infty \tag{23}$$

(ii) *If* $l_1 < 1$, *then*

$$\lim_{k\to\infty}\|\tilde{u}_k\|_\infty \le \frac{m_5}{1 - l_1} \tag{24}$$

and

$$\lim_{k\to\infty}\|\tilde{e}_k\|_\infty \le \left\|\sum_{i=1}^N C_i\right\| \left(\xi_2 \mathrm{e}^{m_2 T} + \left(\frac{m_3 m_5}{m_2(1 - l_1)} + \frac{\epsilon_2}{m_2}\right) \times (\mathrm{e}^{m_2 T} - 1)\right) + \left\|\sum_{i=1}^N D_i\right\| \left(\frac{m_5}{1 - l_1}\right) + \xi_1 \left\|\sum_{i=1}^N (y_i^e + C_i(x_d(t) - x_i^e) + D_i(u_d(t) - u_i^e))\right\|_\infty \tag{25}$$

Proof: From (8), one has

$$\|\tilde{u}_{k+1}(t)\| \le \alpha\|\tilde{u}_k(t)\| + m_1\|\tilde{x}_k(t)\| + \epsilon_1 \tag{26}$$

Furthermore, equation (10) becomes

$$\tilde{x}_k(t) = \tilde{x}_k(0) + \int_0^t (\dot{x}_d(\tau) - \dot{x}_k(\tau))\mathrm{d}\tau \tag{27}$$

which leads to

$$\|\tilde{x}_k(t)\| \le \xi_2 + \int_0^t (m_2\|\tilde{x}_k(\tau)\| + m_3\|\tilde{u}_k(\tau)\| + \epsilon_2)\mathrm{d}\tau \tag{28}$$

which, by virtue of the Bellman–Gronwall Lemma, gives

$$\|\tilde{x}_k(t)\| \le \xi_2 \mathrm{e}^{m_2 t} + \int_0^t (m_3\|\tilde{u}_k(\tau)\| + \epsilon_2)\mathrm{e}^{m_2(t-\tau)}\mathrm{d}\tau \tag{29}$$

Therefore, (26) becomes

$$\|\tilde{u}_{k+1}(t)\| \le \alpha\|\tilde{u}_k(t)\| + m_1\xi_2\mathrm{e}^{m_2 t} + \epsilon_1 + m_1\int_0^t (m_3\|\tilde{u}_k(\tau)\| + \epsilon_2)\mathrm{e}^{m_2(t-\tau)}\mathrm{d}\tau \tag{30}$$

Following the first part of the proof of Theorem 1, the following result is obtained

$$\|\tilde{u}_{k+1}(t)\|_\infty \le \left(\sup_{t\in[0,T]}\{\alpha\} + m_4\right)\|\tilde{u}_k(t)\|_\infty + m_5 \tag{31}$$

Hence, if there exist G_i, $i = 1, \dots, N$, such that $l_1 = \sup_{t\in[0,T]}\{\alpha\} + m_4 < 1$, one has

$$\lim_{k\to\infty}\|\tilde{u}_k(t)\|_\infty \le \frac{m_5}{1 - l_1} \tag{32}$$

Finally, using (6), (29) and (32), we obtain the tracking error bounds given in (25).

Following the second part of the proof of Theorem 1, we obtain

$$\|\tilde{u}_{k+1}(t)\|_\lambda \le \left(\sup_{t\in[0,T]}\{\alpha\} + \frac{m_1 m_3}{\lambda - m_2}(1 - \mathrm{e}^{(m_2-\lambda)T})\right) \times \|\tilde{u}_k(t)\|_\lambda + m_6 \tag{33}$$

Hence, if $\alpha < 1$, there exists $\lambda > m_2 + m_1 m_3$ such that

$$l_2 = \sup_{t\in[0,T]}\{\alpha\} + \frac{m_1 m_3}{\lambda - m_2}(1 - \mathrm{e}^{(m_2-\lambda)T}) < 1$$

Therefore, the following holds

$$\lim_{k\to\infty}\|\tilde{u}_k(t)\|_\lambda \le \frac{m_6}{1 - l_2}$$

which, in view of the definition of the λ-norm, implies that

$$\lim_{k\to\infty}\|\tilde{u}_k(t)\|_\infty \le \mathrm{e}^{\lambda T}\frac{m_6}{1 - l_2} \tag{34}$$

Using (6), (29) and (34), we obtain the tracking error bounds given in (23). □

Now let us assume that the output of system (1) is given by $y_k(t) = Cx_k(t)$ and consider the iterative control law

$$u_{k+1}(t) = u_k(t) + \left(\sum_{i=1}^N G_i\rho_i(z_k)\right)\dot{e}_k(t) \tag{35}$$

where $\dot{e}_k(t) = \dot{y}_d(t) - \dot{y}_k(t)$.

Following the same development as in the proof of Theorem 1, we get the following expression for the control error at iteration $k+1$

$$\tilde{u}_{k+1}(t) = \tilde{u}_k(t) - \left(\sum_{i=1}^N G_i\rho_i(z_k)\right)\dot{e}_k(t) \tag{36}$$

The time-derivative of the tracking error is given by

$$\begin{aligned}\dot{e}_k(t) &= C(\dot{x}_d(t) - \dot{x}_k(t))\\ &= C\sum_{i=1}^N (f(x_i^e, u_i^e) + A_i(x_d - x_i^e)\\ &\quad + B_i(u_d - u_i^e))\rho_i(z_d)\\ &\quad - C\sum_{i=1}^N (f(x_i^e, u_i^e) + A_i(x_k - x_i^e)\\ &\quad + B_i(u_k - u_i^e))\rho_i(z_k)\\ &= C\sum_{i=1}^N (A_i\tilde{x}_k + B_i\tilde{u}_k))\rho_i(z_k)\\ &\quad + C\sum_{i=1}^N (f(x_i^e, u_i^e) + A_i(x_d - x_i^e)\\ &\quad + B_i(u_d - u_i^e))(\rho_i(z_d) - \rho_i(z_k))\end{aligned} \tag{37}$$

Hence, equation (36) becomes

$$\begin{aligned}\tilde{u}_{k+1}(t) = &\left(I - \sum_{i=1}^N G_i\rho_i(z_k)C\sum_{i=1}^N B_i\rho_i(z_k)\right)\tilde{u}_k(t)\\ &- \left(\sum_{i=1}^N G_i\rho_i(z_k)\right)C\left(\sum_{i=1}^N A_i\rho_i(z_k)\right)\tilde{x}_k(t)\\ &- \left(\sum_{i=1}^N G_i\rho_i(z_k)\right)C\sum_{i=1}^N (f(x_i^e, u_i^e)\\ &+ A_i(x_d - x_i^e) + B_i(u_d - u_i^e))(\rho_i(z_d) - \rho_i(z_k))\end{aligned} \tag{38}$$

Since $\sum_{i=1}^N \rho_i(z_k) = 1$, from the latter one can obtain the inequality

$$\|\tilde{u}_{k+1}(t)\| \le \alpha^* \|\tilde{u}_k(t)\| + \|C\| \left\| \sum_{i=1}^{N} G_i \right\| \left\| \sum_{i=1}^{N} A_i \right\| \|\tilde{x}_k(t)\| + \|C\| \left\| \sum_{i=1}^{N} G_i \right\| \sum_{i=1}^{N} \|f(x_i^e, u_i^e) + A_i(x_d - x_i^e) + B_i(u_d - u_i^e)\| \|\rho_i(z_d) - \rho_i(z_k)\| \tag{39}$$

where

$$\alpha^* = \left\| I - \sum_{i=1}^{N} G_i \rho_i(z_k) C \sum_{i=1}^{N} B_i \rho_i(z_k) \right\|$$

Following the steps of the proof of Theorem 1, we obtain

$$\sup_{t\in[0,T]} \|\tilde{u}_k(t)\| \le \left(\sup_{t\in[0,T]} \{\alpha^*\} + \beta^* \right)^k \sup_{t\in[0,T]} \|\tilde{u}_0(t)\| \tag{40}$$

and in terms of the λ-norm

$$\|\tilde{u}_k(t)\|_\lambda \le \left(\sup_{t\in[0,T]} \{\alpha^*\} + \frac{\psi^*}{\lambda - a}(1 - \mathrm{e}^{(a-\lambda)T}) \right)^k \|\tilde{u}_0(t)\|_\lambda \tag{41}$$

where $\psi^* = \mathrm{abg}\|C\|$ and $\beta^* = \psi^*(e^{aT} - 1)/a$. The conclusion of the previous development can be summarized in the following theorem.

Theorem 3: *Consider system* (1) *with the output* $y_k = Cx_k$ *and the ILC* (35). *If Assumptions A1–A3 are fulfilled, then, for all* $t \in [0, T]$, *the following hold:*

(i) *If* $\sup_{t\in[0,T]} \{\alpha^*\} < 1$, *there exists* $\lambda > a + \psi^*$ *such that* $\|y_d(t) - y_k(t)\|_\infty$ *tends to zero when k tends to infinity, with a rate of convergence less than or equal to*

$$\gamma_1 = \mathrm{e}^{\lambda T} \left(\sup_{t\in[0,T]} \{\alpha^*(t)\} + \frac{\psi^*}{\lambda - a}(1 - \mathrm{e}^{(a-\lambda)T}) \right)^k$$

(ii) *If* $\sup_{t\in[0,T]} \{\alpha^*(t)\} + \beta^* < 1$, *then* $\|y_d(t) - y_k(t)\|_\infty$ *converges to zero when k tends to infinity, with a rate of convergence less than or equal to*

$$\gamma_2 = \left(\sup_{t\in[0,T]} \{\alpha^*(t)\} + \frac{\psi^*}{a}(\mathrm{e}^{aT} - 1) \right)^k$$

Now, some remarks should be pointed out.

Remarks:

(1) Note that the parameter λ is not needed in the design of the learning controller. This parameter can be viewed as an indicator for the rate of convergence of the learning process. It is evident that if λT is large, then $\|\tilde{u}_k(t)\|_\lambda << \|\tilde{u}_k(t)\|_\infty$, and the number of necessary iterations to achieve accurate tracking will increase with λT. Therefore, the convergence rates obtained with the λ-norm (case (i) of Theorems 1–3) will be much lower than those obtained with the infinity norm (case (ii) of Theorems 1–3). However, the conditions obtained with the infinity norm are more restrictive than those obtained with the λ-norm in terms of the system dynamics knowledge and the dependence on the tracking horizon.

(2) In case (i) of Theorems 1 and 2, the sufficient condition for the tracking error is $\sup_{t\in[0,T]}\{\alpha(t)\} < 1$. This condition is related to matrices D_i, while matrices A_i, B_i and C_i are not needed. In case (i) of Theorem 3, only matrices B_i and C have to be known. This demonstrates the robustness of the tracking controller, which needs only partial knowledge of the system.

(3) The parameter α depends on the validity functions, which are not constant. This fact makes the choice of the control parameters G_i quite difficult. Nevertheless, if we assume that for any $t \in [0, T]$, there exists a unique model i such that $\rho_i \to 1$, one can conclude that $\rho_j \to 0$ whenever $i \neq j$. Under this assumption, the condition $\sup_{t\in[0,T]}\{\alpha(t)\} < 1$ is fulfilled in Theorems 1 and 2 as long as

$$\|I - G_i D_i\| < 1, \qquad i = 1, \ldots, N \tag{42}$$

and in Theorem 3, as long as

$$\|I - G_i C B_i\| < 1, \qquad i = 1, \ldots, N \tag{43}$$

4. Numerical example

In order to demonstrate the effectiveness of the proposed control scheme, we consider the following non-affine-in-input system

$$\left.\begin{aligned} \dot{x} &= 0.5x + u^2 + (x+1)u \\ y &= x \end{aligned}\right\} \tag{44}$$

where $y \in \mathbb{R}$ is the output and $u \in \mathbb{R}$ is the control input. The objective of the control is to track the trajectory $y_d(t) = 1 - \mathrm{e}^{-t}$, over the time interval $[0, 6]$.

i	(x_i^e, u_i^e)	a_i	b_i
1	(0.05, −1.0256)	−0.5256	−1.0012
2	(0.1, −1.0525)	−0.5525	−1.0050
3	(0.3, −1.1720)	−0.6720	−1.0440
4	(0.4, −1.2385)	−0.7385	−1.0770
5	(0.5, −1.3090)	−0.8090	−1.1180
6	(0.6, −1.3831)	−0.8831	−1.1662
7	(0.8 , −1.5403)	−1.0403	−1.2806
8	(1, −1.7071)	−1.2071	−1.4142

Table 1. Local models

As the first step of the design process, we linearize the system about eight equilibrium points (x_i^e, u_i^e) describing the desired trajectory $y_d(t)$ to obtain eight local models of the form

$$\dot{x} = a_i x + b_i u, \qquad i = 1, 8 \tag{45}$$

where a_i and b_i are given in table 1.

System (44) can be approximated by

$$\left.\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{8} (a_i(x(t) - x_i^e) + b_i(u(t) - u_i^e))\rho_i(z(t)) \\ y(t) &= x(t) \end{aligned}\right\} \tag{47}$$

where $z(t) = [x(t), u(t)]^{\mathrm{T}}$ and $\rho_i(z(t))$ is taken as in (2), with $\Sigma_i = 1$.

The control gains G_i, $i = 1, 8$, are chosen such that $|1 - G_i b_i| < 1$. A simple choice to satisfy this condition is to take $G_i = 1/b_i$, $i = 1, 8$. Note that the parameters a_i are not needed anywhere in our control design.

By applying the ILC scheme (35) to system (44) we obtained the results shown in figures 1 and 2. Figure 1 illustrates the evolution of the Sup-norm of the tracking error with respect to the number of iterations. One can see that we achieve 'almost perfect' tracking at the 13th iteration. Figure 2 shows the time-evolution of the desired trajectory $y_d(t)$ (star-curve) and the output $y_k(t)$ (solid curve) for $k = 5$, $k = 10$ and $k = 15$.

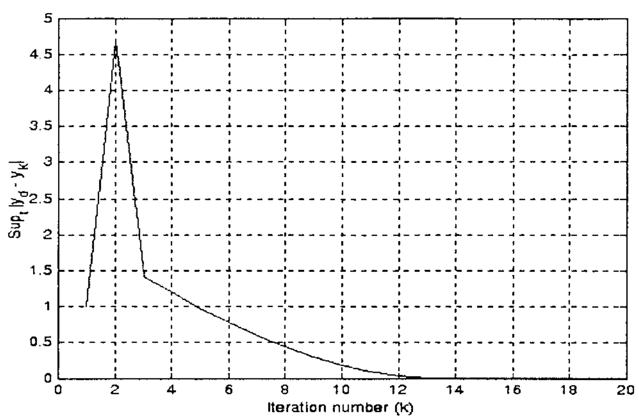

Figure 1. Sup-norm of the tracking error versus the number of iterations.

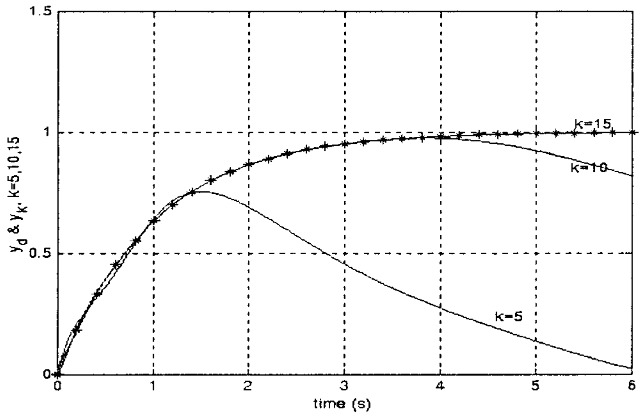

Figure 2. Desired trajectory $y_d(t)$ (star), and the output $y_k(t)$ (solid), for $k = 5, 10$ and 15.

5. Conclusion

We have proposed two iterative learning controllers for continuous-time non-linear systems described by a BMMR with and without direct input transmission. The structures of our controllers are, in fact, those of a P-type and a D-type ILCs with a systematic gain scheduling as a function of the actual operating point. The convergence of the infinity norm and the λ-norm of the tracking error are investigated in the presence of an initial state error (Theorem 2) and in the ideal case (Theorems 1 and 3). It is shown that the rates of convergence obtained with the infinity norm are better than those obtained with the λ-norm. However, the conditions obtained with the infinity norm are more restrictive than those obtained with the λ-norm.

It is well known that the BMMR can be viewed as a quasi-global approximation for non-linear systems including the class of non-affine-in-input systems. As in most approximation problems, the issue of the approximation error has to be addressed. Of course an approximation error will always exist for a finite number of models N, and the controller derived for the BMMR will probably not achieve a 'perfect' tracking when applied to the real non-linear system. The quantification of the tracking error with respect to the number of local models is not an easy task. Nevertheless, under an assumption that the approximation error can be quantified as a Lipschitz function in x, the error can be fairly simply reduced through learning, which is easily observed from equations (6)–(9). If the approximation error is not a Lipschitz function in x or if it is a non-affine function in u, it is difficult to prove the convergence to zero of the tracking error. One can only prove that the tracking error is bounded if the approximation error is bounded.

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