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Invariant Manifold Approach for the Stabilization of Nonholonomic Chained Systems: Application to a Mobile Robot

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Abstract. In this paper it is shown that a class of n-dimensional nonholonomic chained systems can be stabilized using the invariant manifold approach. First, we derive an invariant manifold for this class of systems and we show that, once on it, all the closed-loop trajectories tend to the origin under a linear smooth time-invariant state feedback. Thereafter, it is shown that this manifold can be made attractive by means of a discontinuous time-invariant state feedback. Finally, a mobile robot is taken as an example demonstrating the effectiveness of our study.

Keywords: Nonlinear control systems, feedback stabilization, invariant manifold technique, nonholonomic chained systems, mobile robots.

1. Introduction

Wheeled mobile robots are widely involved in advanced applications such as planetary exploration, intervention in hostile and dangerous environments, and the execution of accurate repetitive tasks. Unfortunately, this class of mechanical systems is subject to nonholonomic kinematic constraints restricting its mobility and making its control non-trivial. The problem of controlling nonholonomic systems has been tackled following two principal approaches [6]. The first approach uses open-loop control strategies to generate feasible trajectories; it is often referred to as motion planning [8–10]. Although this approach is interesting from the practical point of view (obstacles avoidance for example), it suffers from a lack of robustness with respect to disturbances and modeling inaccuracies. The second approach uses feedback control strategies, guaranteeing a certain level of robustness, to solve the following two fundamental problems:

1. Path-following,
2. Point-stabilization (or stabilization).

The first problem consists in finding adequate feedback control laws allowing the mobile robot to track a desired trajectory. The second problem consists in finding adequate feedback control laws allowing the mobile robot to reach a desired static configuration starting from any initial configuration.

It is well known that the first problem can be solved by means of classical time-invariant smooth state feedback [13], while the second one needs more elaborate nonlinear techniques. The second problem is a challenging one, since it has been proven that nonholonomic systems cannot be stabilized to an equilibrium point via any smooth time-invariant feedback [3]. To overcome this problem, two main types of controller have been proposed in the literature:

1. Smooth time-varying controllers leading to slow (polynomial) asymptotic convergence (see, for instance, [11, 14]).
2. Discontinuous time-invariant controllers leading to exponential convergence and generating more realistic trajectories, as shown in [1, 2, 4, 15].

In this paper we aim to provide another way to solve the point-stabilization problem for n-dimensional nonholonomic chained systems by means of a time-invariant discontinuous state feedback. To this end, the invariant manifold method, previously used for low order nonholonomic systems [5, 12, 17], is investigated and generalized to the n-dimensional nonholonomic chained systems, as shown in [16].

The invariant manifold approach consists in finding a suitable manifold, which is invariant under a linear state feedback, on which all the closed-loop trajectories tend to the origin. Furthermore, this manifold is rendered attractive by adding a discontinuous state feedback to the first linear state feedback to ensure that all the closed-loop trajectories tend to the invariant manifold as long as they start outside a certain manifold.

The next section describes the problem we are concerned with as well as the construction of the invariant manifold. Section 3, is devoted to the design of the discontinuous state feedback making the invariant manifold attractive. In Section 4, our control law is applied to a mobile robot and simulation results are given. Section 5 concludes the paper.

2. Problem Formulation and Invariant Manifold Construction

2.1. Problem Formulation

The problem that we address in this paper is the stabilization of the class of nonholonomic systems in chained form given by

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= = u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1, \end{aligned} \tag{1}$$

where $x = (x_1 x_2 \ldots x_n)^T \in D_1$ denotes the state vector and $(u_1 u_2)^T \in D_2$ denotes the input vector, and D_1 and D_2 are open subsets of $\mathcal{R}^n$ and $\mathcal{R}^2$. Such a class of nonholonomic systems has been widely studied in the literature. Sufficient conditions under which any nonholonomic mechanical system with two inputs, can be transformed, via coordinates and feedback transformations, into a chained form, are given in [10].

Let us first, recall the following definitions.

DEFINITION 1. Let $\Phi : \mathcal{R}^n \to \mathcal{R}^p$ be a smooth map. A manifold $M = \{x \in \mathcal{R}^n / \Phi(x) = 0\}$ is said to be invariant for the control system $\dot{x} = f(x, u)$ if all system trajectories starting in M at $t = t_0$ remain in this manifold for all $t \geq t_0$. In other words, the Lie derivative of Φ along the vector field f is zero ($L_f \Phi(x) = 0$) for all $x \in M$.

DEFINITION 2. A manifold $M = \{x \in \mathcal{R}^n / \Phi(x) = 0\}$ is said to be asymptotically attractive in an open domain Ω of $\mathcal{R}^n$ if for all $t_0 \in \mathcal{R}_+$ such that $x(t_0) \in \Omega$, then $\lim_{t\to\infty} x(t) \in M$.

Our aim is to design a discontinuous time-invariant state feedback controller $(u_1(x), u_2(x))^T$ which stabilizes system (1), where

$$u_1(x) = \nu_1(x), \quad u_2(x) = \nu_2(x) + w_2(x). \tag{2}$$

2.2. Construction of the Invariant Manifold

First, let $w_2(x) = 0$, and consider a linear state feedback $(\nu_1(x)\nu_2(x))^T$ such that

$$u_1(x) = -k_1 x_1, \quad u_2(x) = -k_2 x_1 - k_3 x_2, \tag{3}$$

where $k_2 \in \mathcal{R}$, k_1, and k_3 are strictly positive parameters, and $k_1 \neq k_3$.

The resulting closed-loop system (1–3) can be integrated, step by step, to obtain

$$\begin{aligned}
x_1(t) &= x_{10} \exp(-k_1 t), \\
x_2(t) &= \left(x_{20} - \frac{k_2 x_{10}}{K_a}\right) \exp(-k_3 t) + \frac{k_2 x_{10}}{K_a} \exp(-k_1 t), \\
x_3(t) &= \frac{k_1}{K_b} x_{10} \left(x_{20} - \frac{k_2 x_{10}}{K_a}\right) \exp(-K_b t), \\
&\quad + \frac{k_2}{2K_a} x_{10}^2 \exp(-2k_1 t) + S_3(x_0), \\
x_4(t) &= \frac{k_1^2 x_{10}^2}{K_b(k_1 + K_b)} \left(x_{20} - \frac{k_2 x_{10}}{K_a}\right) \exp(-(k_1 + K_b)t) \\
&\quad + \frac{k_2 x_{10}^3}{6K_a} \exp(-3k_1 t) + x_{10} S_3(x_0) \exp(-k_1 t) + S_4(x_0), \\
&\;\;\vdots \\
x_n(t) &= \frac{k_1^{n-2} x_{10}^{n-2}}{K_b(K_b + k_1) \cdots (K_b + (n-3)k_1)} \left(x_{10} - \frac{k_2 x_{10}}{K_a}\right) \exp(-(K_b + (n-3)k_1)t) \\
&\quad + \frac{k_2 x_{10}^{n-1}}{(n-1)! K_a} \exp(-(n-1)k_1 t) \\
&\quad + \sum_{i=1}^{n-3} \frac{x_{10}^i}{i!} S_{n-i}(x_0) \exp(-k_1 i t) + S_n(x_0),
\end{aligned} \tag{4}$$

where $K_a = k_1 - k_3$, $K_b = k_1 + k_3$, and $S_3(x_0)$, $S_4(x_0), \ldots S_n(x_0)$ are the integration constants which can be determined, at $t = 0$, as a function of the initial conditions $x_{10}, x_{20}, \ldots, x_{n0}$.

Besides, from (4), one can easily see that $(x_1, x_2, x_3, x_4, \ldots, x_n)$ tends to $(0, 0, S_3(x_0), S_4(x_0), \ldots, S_n(x_0))$ when t goes to infinity. So, if we take the initial conditions such that $S_j(x_0) = 0;\ 3 \le j \le n$, then the whole state tends to the origin. Let us define $S_j = S_j(x);\ 3 \le j \le n$ such that the initial conditions are all satisfied (i.e., setting $t = 0$ in (4) and determining $S_j(x_0)$ and then substituting x_0 by x).

$$
\begin{aligned}
S_3(x) &= x_3 - \frac{k_1}{K_b}x_1x_2 + \frac{k_2}{2K_b}x_1^2,\\
S_4(x) &= x_4 - x_1x_3 + \frac{k_1}{(k_1+K_b)}x_1^2x_2 - \frac{k_2}{3(k_1+K_b)}x_1^3,\\
S_5(x) &= x_5 - x_1x_4 + \frac{1}{2}x_1^2x_3 - \frac{k_1}{2(2k_1+K_b)}x_1^3x_2 + \frac{k_2}{8(2k_1+K_b)}x_1^4,\\
&\vdots\\
S_n(x) &= x_n + \sum_{i=1}^{n-3}\frac{(-1)^i x_1^i x_{n-i}}{i!} + \frac{(-1)^n k_1 x_1^{n-2}}{(K_b+(n-3)k_1)(n-3)!}\\
&\quad + \frac{(-1)^{n-1}k_2x_1^{n-1}}{(K_b+(n-3)k_1)(n-3)!(n-1)}.
\end{aligned}
\tag{5}
$$

From (5), it appears that if the state variables belong to the two-dimensional manifold $M_s = \{x \in \mathcal{R}^n / S_i(x) = 0;\ 3 \le i \le n\}$, then the whole state x tends to the origin, since x_1 and x_2 decay exponentially to zero. Furthermore, this manifold is invariant under the linear state feedback ($v_1 = -k_1x_1$, $v_2 = -k_2x_1 - k_3x_2$), as is shown in the following result.

PROPOSITION 1. *Consider the following smooth functions:*

$$
\begin{aligned}
S_j(x) &= x_j + \sum_{i=1}^{j-3}\frac{(-1)^i x_1^i x_{j-i}}{i!} + \frac{(-1)^j k_1 x_1^{j-2}\, x_2}{(K_b+(j-3)k_1)(j-3)!}\\
&\quad + \frac{(-1)^{j-1}\, k_2x_1^{j-1}}{(K_b+(j-3)k_1)(j-3)!(j-1)}, \quad \text{for} \quad j \ge 3.
\end{aligned}
$$

Then, $M_s = \{x \in \mathcal{R}^n / S_j(x) = 0;\ 3 \le j \le n\}$ is an invariant manifold for the closed-loop system (1–3).

Proof. Evaluating the Lie derivatives of S_j along the vector fields of system (1) under the linear state feedback (3), yields

$$
\begin{aligned}
\dot{S}_j(x) &= L_f S_j = \frac{(-1)^{j-1}\, x_1^{j-3}}{(j-3)!(K_b+(j-3)k_1)}((k_3x_2 + k_2x_1)u_1 - k_1x_1u_2) \equiv 0\\
&\quad \text{for all} \quad x \in \mathcal{R}^n \quad \text{and} \quad 3 \le j \le n.
\end{aligned}
$$

□

Two remarks should be pointed out. Firstly, the transformation from $(x_1, x_2, \ldots, x_n)$ to $(x_1, x_2, S_3, \ldots, S_n)$ is a diffeomorphism. Hence, the stabilization of the chained system (1) is equivalent to the stabilization of $(x_1, x_2, S_3, \ldots, S_n)$. Secondly, the invariance of the manifold M_s means that once on it, the trajectories of the closed-loop system remain there for all subsequent times. Hence, to stabilize system (1), it suffices to bring the state variables $(x_1, x_2, \ldots, x_n)$ into M_s by an additional state feedback, namely, $w_2(x)$.

3. Discontinuous State Feedback Control Synthesis

Now, we focus our objective in the determination of the second term of (2), namely $w_2(x)$, making the manifold M_s attractive. Once on it, $w_2(x)$ vanishes and the whole state $(x_1, x_2, \ldots, x_n)$ tends to zero under the residual linear state feedback (3).

Simple algebraic manipulations can be used to show that

$$\begin{aligned}
\dot{x}_1 &= u_1,\\
\dot{x}_2 &= u_2,\\
\dot{S}_3 &= \frac{1}{K_b}((k_3x_2 + k_2x_1)u_1 - k_1x_1u_2),\\
\dot{S}_4 &= \frac{-x_1}{(k_1 + K_b)}((k_3x_2 + k_2x_1)u_1 - k_1x_1u_2),\\
\dot{S}_5 &= \frac{x_1^2}{2(2k_1 + K_b)}((k_3x_2 + k_2x_1)u_1 - k_1x_1u_2),\\
&\vdots\\
\dot{S}_n &= \frac{(-1)^{n-1}\; x_1^{n-3}}{(n-3)!((n-3)k_1 + K_b)}((k_3x_2 + k_2x_1)u_1 - k_1x_1u_2).
\end{aligned} \tag{6}$$

Under the control law (2), system (6) becomes

$$\begin{aligned}
\dot{x}_1 &= -k_1x_1,\\
\dot{x}_2 &= -k_2x_1 - k_3x_2 + w_2,\\
\dot{S}_3 &= \frac{-k_1x_1}{K_b}w_2,\\
\dot{S}_4 &= \frac{k_1x_1^2}{(k-1 + K_b)}w_2,\\
\dot{S}_5 &= \frac{-k_1x_1^3}{2(2k_1 + K_b)}w_2,\\
&\vdots\\
\dot{S}_n &= \frac{(-1)^n\; k_1x_1^{n-2}}{(n-3)!((n-3)k_1 + K_b)}w_2.
\end{aligned} \tag{7}$$

Let us define the following vectors and matrices that will be used later:

$$S = \begin{pmatrix} S_3 \\ S_4 \\ \vdots \\ S_n \end{pmatrix}, \quad C = \begin{pmatrix} c_3 \\ c_4 \\ \vdots \\ c_n \end{pmatrix}, \quad A = \begin{pmatrix} k_1 & 0 & \cdots & & 0 \\ 0 & 2k_1 & & & \\ & & \ddots \ddots & & \\ \vdots & & \ddots \ddots & & \vdots \\ & & & & \ddots 0 \\ 0 & & \cdots & & 0\ (n-2)k_1 \end{pmatrix},$$

$$B = \begin{pmatrix} b_3 \\ b_4 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \frac{-k_1}{K_b} \\ \frac{k_1}{k_1+K_b} \\ \vdots \\ \frac{(-1)^n\ k_1}{(n-3)!(k_b+(n-3)k_1)} \end{pmatrix}$$

and .

$$Q(x_1) = \begin{pmatrix} \frac{1}{x_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{x_1^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{x_1^{n-2}} \end{pmatrix}.$$

We are now ready to state the following result:

PROPOSITION 2. *The invariant manifold* $M_s = \{x \in \mathcal{R}^n / S_i(x) = 0;\ 3 \le i \le n\}$ *is attractive, over the domain* $\Omega = \{x = (x_1, x_2, \ldots, x_n) \in \mathcal{R}^n / x_1 \neq 0\}$*, under the following control law:*

$$w_2 = C^T Q(x_1) S, \tag{8}$$

where the vector C *is chosen such that the eigenvalues of the matrix* $(A + BC^T)$ *are all with a negative real part.*

Proof. Since the vector C is such that the eigenvalues of the matrix $(A + BC^T)$ are all with a negative real part, then there exists a symmetric positive definite matrix P satisfying

$$(A + BC^T)^T\ P + P(A + BC^T) < 0.$$

Now, consider the following Lyapunov function candidate, which is positive definite over Ω and outside M_s:

$$V(S, x_1) = (QS)^T\ P(QS). \tag{9}$$

Denoting $Y = QS$, and differentiating (9) with respect to time leads to

$$\dot{V}(S, x_1) = \dot{Y}^T PY + Y^T P\dot{Y}. \tag{10}$$

Using some algebraic manipulations, one can show that $\dot{Y} = AY + Bw_2$ and, hence, (10) becomes

$$\dot{V}(S, x_1) = (AY + Bw_2)^T \ PY + Y^T P(AY + Bw_2). \tag{11}$$

Under the control law (8), the latter becomes

$$\dot{V}(S, x_1) = (QS)^T((A + BC^T)^T \ P + P(A + BC^T))(QS), \tag{12}$$

which is negative definite outside the manifold M_s and over the domain Ω. Therefore, $V(S, x_1)$ decreases with respect to time and tends to zero when t tends to infinity. Thus, $||Q(x_1)S||$ is bounded and tends to zero. Therefore, the convergence of S to zero becomes obvious, since x_1 decays exponentially to zero. □

If we examine system (1), one can easily see that if x_1 converges to zero before $x_3, x_4, \ldots, x_n$ then the convergence of the whole state to the origin never holds, which is the main problem of nonholonomic systems. Intuitively, one must ensure that the convergence of the whole state towards the invariant manifold M_s is faster than the convergence of x_1 to zero, which justifies the choice of the weighting matrix $Q(x_1)$.

Now, one can state our main result in the following theorem:

THEOREM 1. *Consider system* (1) *under the following control law:*

$$u_1 = -k_1x_1, \quad u_2 = -k_2x_1 - k_3x_2 + w_2, \tag{13}$$

where $w_2 = C^T Q(x_1)S, k_2 \in \mathcal{R}, k_1 > 0, k_3 > 1$ *and* $k_1 \neq k_3$.

The vector C *is such that the eigenvalues of the matrix* $(A + BC^T)$ *are all with negative real parts. Then, if* $x_1(0) \neq 0$,

(i) *the whole state of the closed loop system* (1–13) *tends to zero when* t *tends to infinity;*
(ii) *the control law* (13) *is well defined and bounded for all* $t \geq 0$.

Proof. (i) Recall that the transformation from $(x_1, x_2, \ldots, x_n)$ to $(x_1, x_2, S_3, \ldots, S_n)$ is a diffeomorphism and hence, the stabilization of the chained system (1) is equivalent to the stabilization of system (6).

Thanks to Proposition 2, which proves the convergence of S to zero and the boundedness of $||QS||$. Thanks also to Proposition 1, which proves that if the manifold M_s is reached, all the trajectories of the closed-loop system remain there. These imply that x_1 and x_2 tend exponentially to zero under the residual linear state feedback $(u_1 = -k_1x_1, u_2 = -k_2x_1 - k_3x_2)$, since $C^T Q(x_1)S \to 0$ when $t \to \infty$, as long as x belongs to Ω. Finally, one can conclude that the whole state x tends to zero when $t \to \infty$.

(ii) Now one has to show that the control law is bounded and well defined for all $t \geq 0$ provided that $x_1(0) \neq 0$. Indeed, since the boundedness of $C^T Q(x_1)S$ is proved in Proposition 1, and the boundedness of x_1 is obvious, we have just to prove the boundedness of the state variable x_2.

To this end, let us consider the following Lyapunov candidate function

$$V_T(x_2) = \frac{1}{2}x_2^2. \tag{14}$$

Differentiating (14) with respect to time along the trajectories of the closed loop system (6–13) leads to

$$\dot{V}_T = -k_2 x_1 x_2 - k_3 x_2^2 + x_2 C^T Q(x_1) S = -k_3 x_2^2 + x_2 (C^T Q(x_1) S - k_2 x_1). \tag{15}$$

Now, one can use Young's inequality [7], which states that if the constants $p > 1$ and $q > 1$ are such that $(p-1)(q-1) = 1$, then for all $\varepsilon > 0$ and all $(z_1, z_2) \in \mathcal{R}^2$, we have

$$z_1 z_2 \leq \frac{\varepsilon^p}{p} |z_1|^p + \frac{1}{q \varepsilon^q} |z_2|^q. \tag{16}$$

Choosing $p = q = 2$ and $\varepsilon^2 = 2$, (16) becomes

$$z_1 z_2 \leq z_1^2 + \frac{1}{4} z_2^2. \tag{17}$$

Using this inequality for (15), with $z_1 = x_2$ and $z_2 = C^T Q(x_1) S - k_2 x_1$, we obtain

$$\dot{V}_T \leq -(k_3 - 1) x_2^2 + \frac{1}{4} (C^T Q(x_1) S - k_2 x_1)^2. \tag{18}$$

The term $(C^T Q(x_1) S - k_2 x_1)$ is bounded and tends to zero when t tends to infinity.

From (18), one can see that $\dot{V}_T$ is negative whenever

$$|x_2| > \frac{(C^T Q(x_1) S - k_2 x_1)}{2\sqrt{k_3 - 1}}.$$

Since, $(C^T Q(x_1) S - k_2 x_1)$ is bounded, one can conclude that $\dot{V}_T$ is negative outside the compact residual set

$$\left\{ x_2 / |x_2| \leq \frac{||(C^T Q(x_1) S - k_2 x_1)||_\infty}{2\sqrt{k_3 - 1}} \right\}.$$

In view of (14), $|x_2|$ decreases whenever $x_2(t)$ is outside the previous set and, hence, $x_2(t)$ is bounded as

$$||x_2||_\infty \leq \left\{ |x_2(0)|, \frac{||(C^T Q(x_1) S - k_2 x_1)||_\infty}{2\sqrt{k_3 - 1}} \right\}.$$

Finally, one can conclude that the control law is well defined and bounded as along as $x_1(0) \neq 0$, since x_1 decays to zero without crossing $x_1 = 0$. □

It is worth noting that the assumption $x_1(0) \neq 0$ is not very restrictive, since it is always possible to apply an open loop control, for an arbitrary small period of time, to drive the system away from $x_1 = 0$ and then switch to the state feedback (13).

4. Application to a Car-Like Mobile Robot

The mobile robot under consideration is a car-like mobile robot with a motorized front wheel and two passive rear-wheels (see Figure 1). The motion control of this vehicle can be achieved by dealing with the linear velocity of the point M denoted ν and the steering velocity of the front wheel denoted ω.

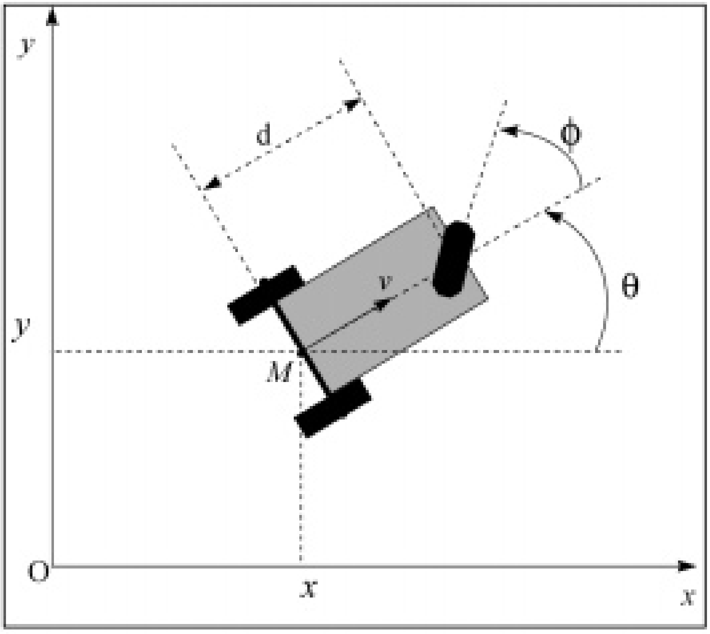

Figure 1. A car-like vehicle.

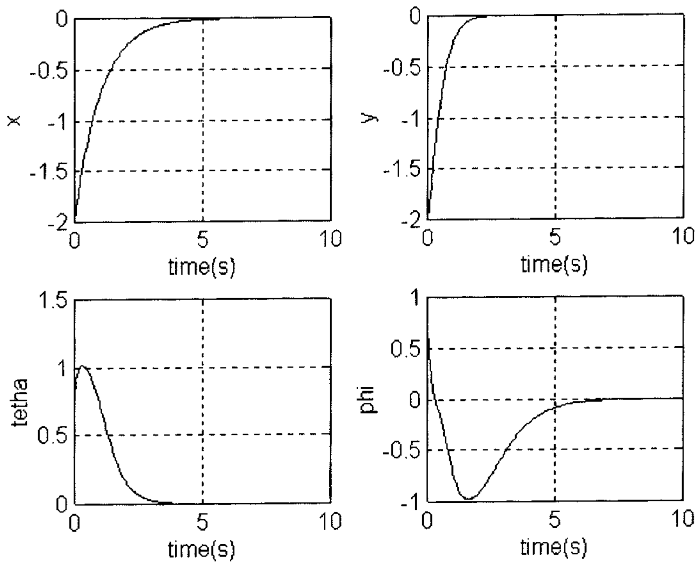

Figure 2. Time evolution of the state variables starting from ($x_0 = -2$, $y_0 = -2$, $\theta_0 = \pi/4$, $\phi_0 = \pi/4$).

The kinematics model is derived under rolling without slippage assumption, on horizontal ground. The configuration of the mobile robot is described by the vector $(x, y, \theta, \phi)^T$, where (x, y) are the coordinates of the point M, located at mid-distance of the rear-wheels, θ is the orientation of the vehicle, taken counterclockwise from the global x-axis, and ϕ is the steering angle of the front wheel.

$$\dot{x} \; = \; v\cos\theta, \quad \dot{y} = v\sin\theta,$$

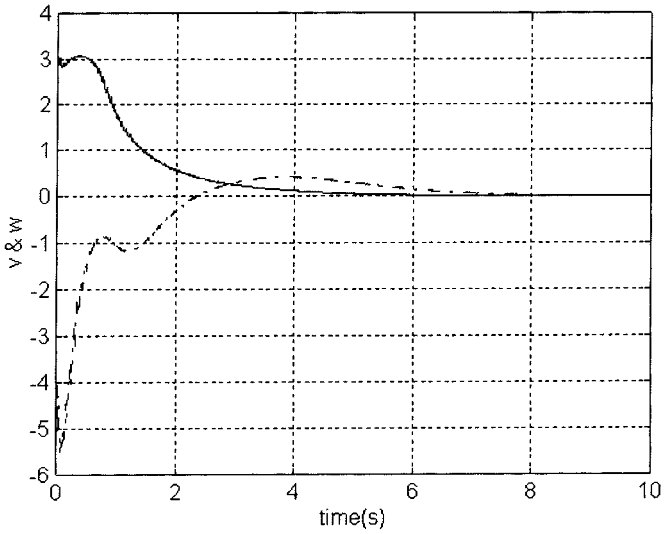

Figure 3. Time plots of the control inputs v (—) and ω (-·-).

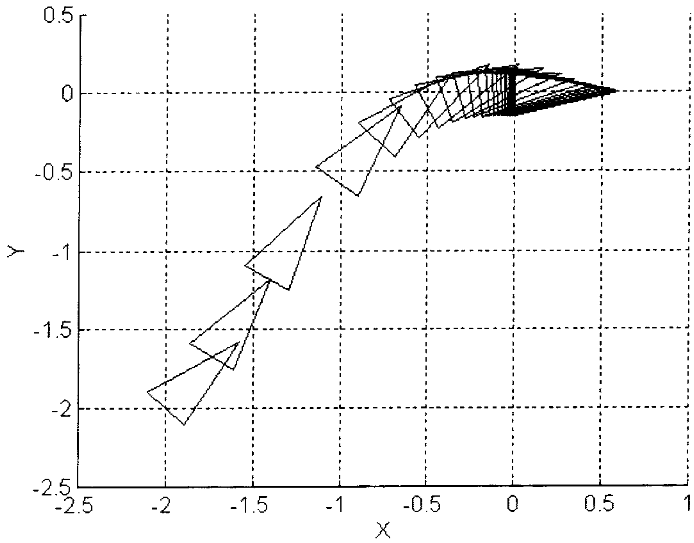

Figure 4. Steering the vehicle to the origin, starting from ($x_0 = -2$, $y_0 = -2$, $\theta_0 = \pi/4$, $\phi_0 = \pi/4$).

$$\dot{\theta} = \frac{v}{d}\tan\phi, \quad \dot{\phi} = \omega, \tag{19}$$

where d denotes the distance between the point M and the center of the front wheel. As discussed in [10], system (19) can be transformed into a four-order chained system using the following local coordinates transformation defined over the subset

$$\Gamma = \left\{(x, y, \theta, \phi) \in \mathcal{R}^4/\theta \neq \frac{\pi}{2} \bmod \pi, \phi \neq \frac{\pi}{2} \bmod \pi\right\},$$

$$\begin{aligned} x_1 &= x, \quad x_2 = \frac{\tan\phi}{d\cos^3\theta}, \\ x_3 &= \tan\theta, \quad x_4 = y, \end{aligned} \tag{20}$$

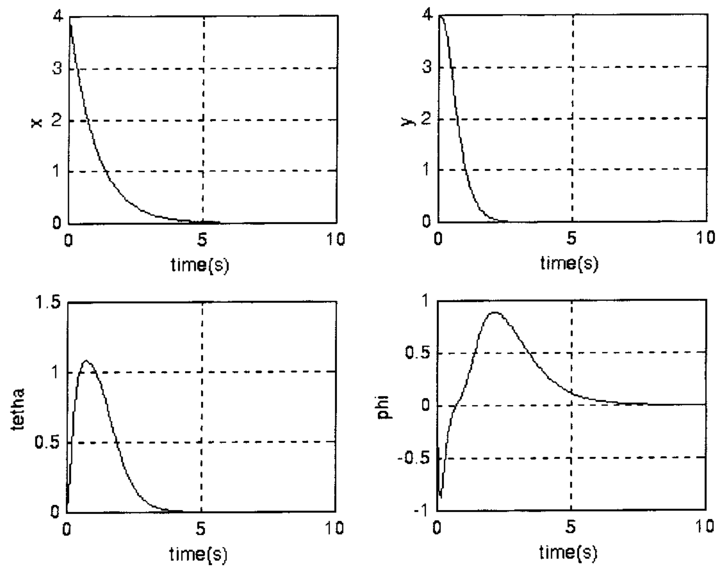

Figure 5. Time evolution of the state variables, starting from ($x_0 = 4$, $y_0 = 4$, $\theta_0 = 0$, $\phi_0 = 0$).

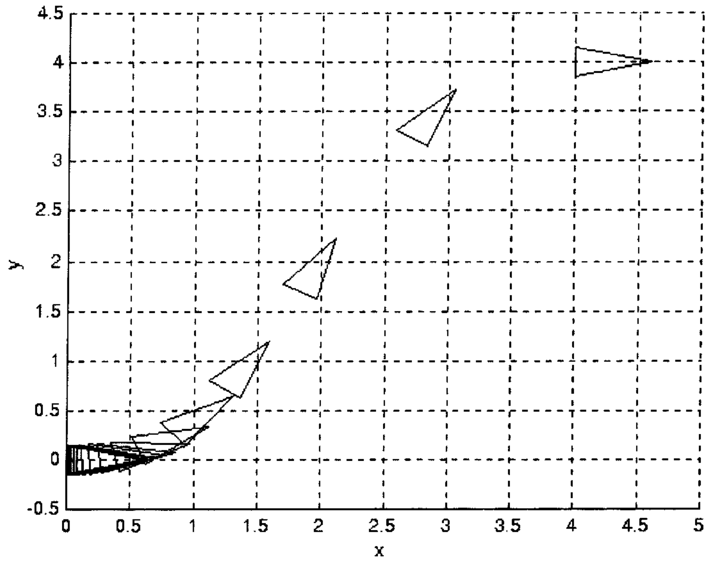

Figure 6. Steering the vehicle to the origin, starting from ($x_0 = 4$, $y_0 = 4$, $\theta_0 = 0$, $\phi_0 = 0$).

and the following input transformation defined over the domain Γ:

$$\begin{aligned} v &= \frac{u_1}{\cos\theta}, \\ \omega &= -\frac{3\sin^2\phi\sin\theta}{d\cos^2\theta}u_1 + d\cos^2\phi\cos^3\theta u_2, \end{aligned} \tag{21}$$

where u_1 and u_2 are the control variables given in (13).

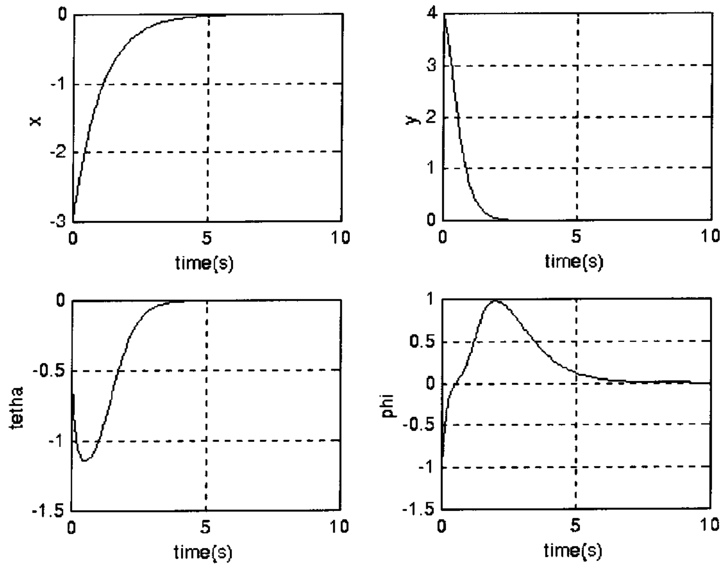

Figure 7. Time evolution of the state variables, starting from ($x_0 = -3$, $y_0 = 4$, $\theta_0 = -\pi/6$, $\phi_0 = -\pi/3$).

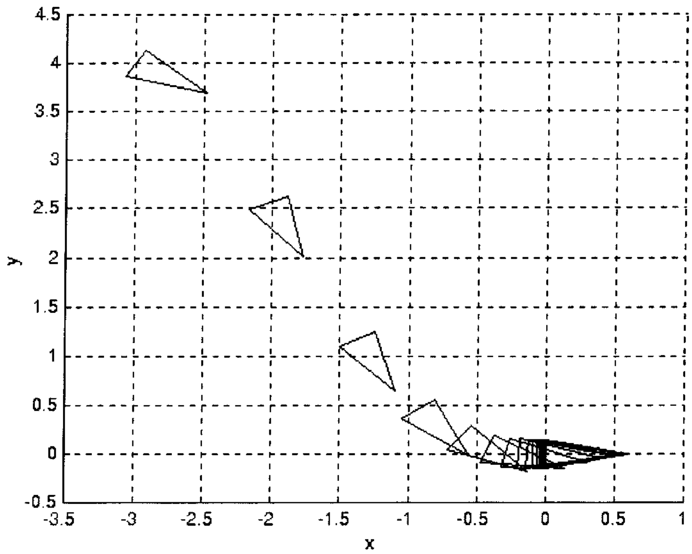

Figure 8. Steering the vehicle to the origin, starting from ($x_0 = -3$, $y_0 = 4$, $\theta_0 = -\pi/6$, $\phi_0 = -\pi/3$).

In our simulations, we have taken $d = 1.2$ m, $k_1 = 1$, $k_2 = 0$, $k_3 = 3$ and $k_b = k_1 + k_3 = 4$. We have also used $S = [S_3 \;\; S_4]^T$ with

$$S_3(x) = x_3 - \frac{k_1}{K_b} x_1 x_2 + \frac{k_2}{2K_b} x_1^2,$$

$$S_4(x) = x_4 - x_1 x_3 + \frac{k_1}{(k_1 + K_b)} x_1^2 x_2 - \frac{k_2}{3(k_1 + K_b)} x_1^3.$$

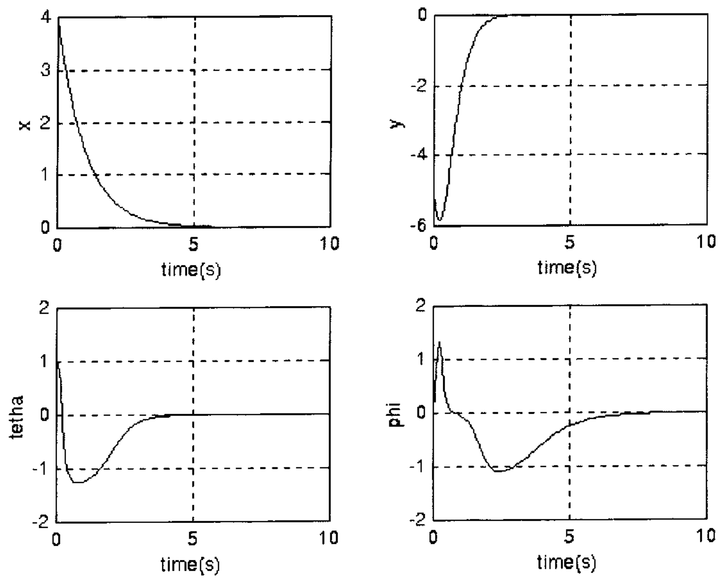

Figure 9. Time evolution of the state variables, starting from ($x_0 = 4$, $y_0 = -5$, $\theta_0 = -\pi/3$, $\phi_0 = 0$).

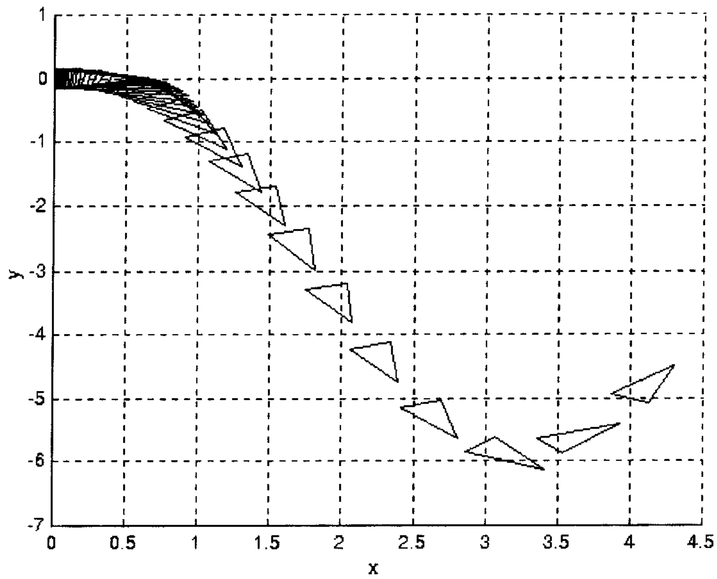

Figure 10. Steering the vehicle to the origin, starting from ($x_0 = 4$, $y_0 = -5$, $\theta_0 = -\pi/3$, $\phi_0 = 0$).

The matrices A, B, and Q are given by

$$A = \begin{bmatrix} k_1 & 0 \\ 0 & 2k_1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{-k_1}{K_b} \\ \frac{k_1}{k_1+K_b} \end{bmatrix}, \quad \text{and} \quad Q(x_1) = \begin{bmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_1^2} \end{bmatrix}.$$

Finally, we have chosen $C^T = [-24 \quad -60]$ to make the eigenvalues of $(A + BC^T)$ equal to -1 and -2.

We have performed some simulations for different initial conditions. The convergence of the whole state of system (19) to the origin starting from ($x_0 = -2$, $y_0 = -2$, $\theta_0 = \pi/4$,

$\phi_0 = \pi/4$), and the time evolution of the control variables are shown, respectively, in Figures 2 and 3. Figure 4 shows the motion of the vehicle towards the origin starting from the same initial conditions. Figures 5, 7 and 9 show the convergence of the whole state of system (19) to the origin for different initial configurations, while Figures 6, 8 and 10 illustrate the motion of the vehicle towards the origin starting from different initial configurations.

5. Conclusion

In this paper, it is demonstrated that the invariant manifold technique can be applied for the stabilization of n-dimensional nonholonomic systems in chained form. In fact, using this technique we have proposed a discontinuous time-invariant state feedback controller for the stabilization of this class of strong nonlinear systems. Finally, our controller has been successfully applied to a car-like mobile robot demonstrating that our control scheme fits particularly well for the stabilization of mobile robots.

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