# Unit Quaternion-Based Output Feedback for the Attitude Tracking Problem

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***Abstract*—In this note, we propose a quaternion-based dynamic output feedback for the attitude tracking problem of a rigid body without velocity measurement. Our approach consists of introducing an auxiliary dynamical system whose output (which is also a unit quaternion) is used in the control law together with the unit quaternion representing the attitude tracking error. Roughly speaking, the necessary damping that would have been achieved by the direct use of the angular velocity can be achieved, in our approach, by the vector part $\tilde{q}$ of the error signal between the output of the auxiliary system and the unit quaternion tracking error. The resulting velocity-free control scheme guarantees *almost* global[1] asymptotic stability which is as strong as the topology of the motion space can permit. In the regulation case, our control law is a pure quaternion feedback (i.e., consisting of two terms that are vector parts of unit-quaternion), and hence, the control torques are naturally bounded by the control gains. Simulation results are provided to show the effectiveness of the proposed control scheme.**

***Index Terms*—Attitude tracking, output feedback, rigid body.**

## I. Introduction

The attitude control problem of a spacecraft, or a rigid body in general, has been extensively studied during the past four decades. This is a particularly interesting problem in dynamics since the angular velocity of the body cannot be integrated to obtain the attitude of the body [8]. From a practical point of view, the design of efficient and low-cost attitude controllers is an important issue which is of great interest for aerospace industry for instance. The attitude stabilization of a rigid body, using the unit-quaternion and the angular velocity in the feedback control law, has been investigated by many researchers and a wide class of controllers has been proposed (see, for instance, [8], [15], [18], [19]). In [15], some quaternion-based feedback controllers for the attitude stabilization have been proposed and tested experimentally on a quadrotor aircraft.

The attitude control of a rigid body with full states measurements (i.e., quaternion and angular velocity), being relatively well understood, the research has been directed towards other performance and implementation-cost optimization issues, by removing the requirement of the velocity measurement. The passivity property, was the main idea behind the design of the attitude controllers, without velocity measurement, in [6], [10], [17]. In fact, in [6], the authors used the passivity-based adaptive control approach for robotic manipulators to derive their adaptive attitude control scheme without velocity measurement. In [10], a quite similar passivity argument has been used to develop a velocity-measurement-free attitude stabilization controller using a lead filter. In [16], an alternative solution to the attitude regulation problem without velocity measurement and without the use of a lead filter has been proposed. The author in [17] derives quite similar results as the results of [10] by using the Rodrigues Parameters instead of the quaternion [13]. The second approach that has also been used to avoid the velocity measurement is based on the use of nonlinear observers. In fact, in [12], a nonlinear velocity observer, using just the torque and orientation measurements, has been proposed based on the analogy to second-order linear systems, where a separation principle-like property was conjectured. The extension of the velocity-free attitude regulation controllers to the tracking problem is not an obvious task especially when we aim for nonlocal results. In [3], two attitude tracking controllers without velocity measurement have been proposed. The first one is a locally exponentially stabilizing controller-observer scheme. The second scheme, guaranteeing also local exponential stability under an adequate choice of the control parameters, is a generalization of the lead filter-based regulation scheme of [10] to the attitude tracking problem. In [4], a local velocity-free adaptive quaternion-based tracking controller for a rigid body with uncertainties has been proposed. Another alternative to the work of [3] has been proposed in [1] based on the results of [17] using the Rodrigues parameters instead of the unit-quaternion. Note that unlike the quaternion representation, the three-parameters (Rodrigues parameters) attitude representation suffers from singularity problems [13].

In the present paper, we use the four-parameters representation (quaternion), which is globally nonsingular, to represent the attitude motion, and provide a new solution to the attitude tracking problem without velocity measurement. To the best of our knowledge, our result is the first velocity-free unit quaternion-based tracking controller guaranteeing *almost* global asymptotic stability. Our main idea is the introduction of an auxiliary unit-quaternion dynamical system having the same structure as the actual unit-quaternion attitude model. Under an appropriate feedback involving the unit quaternion tracking error and the vector part $\tilde{q}$ of the error signal between the output of the auxiliary system and the unit quaternion tracking error, we show that the map between the auxiliary system input and $\tilde{q}$ is passive. Therefore, the auxiliary system input can be designed as a simple proportional feedback in terms of $\tilde{q}$. The proposed control strategy guarantees *almost* global asymptotic attitude tracking. In the regulation case, our control law turns out to be a pure quaternion feedback leading to a natural boundednes of the control torques, and hence, the designer can explicitly set the desired bounds for the control torques through the control gains. Finally, simulation results are also provided to support the theoretical developments.

## II. Dynamical Model and Problem Statement

The dynamical model of a spacecraft or a rigid body is given by

$$I_f \dot{\Omega} = -\Omega \times I_f \Omega + \tau, \tag{1}$$

$$\dot{R} = RS(\Omega) \tag{2}$$

where $\Omega$ denotes the angular velocity of the body expressed in the body-fixed frame $\mathcal{A}$. The orientation of the rigid body is given by the orthogonal rotation matrix $R \in SO(3)$. $I_f \in \mathbb{R}^{3\times 3}$ is a symmetric positive definite constant inertia matrix of the body with respect to the

Manuscript received January 26, 2007; revised July 9, 2007. Current version published August 29, 2008. Recommended by Associate Editor F. Bullo. This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC). A preliminary version of this work, dealing with the regulation case, has been presented at the 45th IEEE Conference on Decision and Control, San Diego, CA, 2006.

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Digital Object Identifier 10.1109/TAC.2008.927789

[1]In this paper we use the term *almost* global (see, for instance, [9]) to indicate that the boundedness of the states is guaranteed over $\mathbb{Q}_u \times \mathbb{Q}_u \times \mathbb{R}^3$, where $\mathbb{Q}_u$ is the set of unit-quaternion. Furthermore, the closed-loop system has four equilibrium points (three repeller equilibria and one attractor) that are mathematically different but represent the same physical attitude of the rigid body. All trajectories starting in $\mathbb{Q}_u \times \mathbb{Q}_u \times \mathbb{R}^3$–except the three repeller equilibria—will converge to the unique attractor equilibrium. For more details about the topological obstruction to continuous global stabilization of rotational motion, the reader is referred to [2].

frame $\mathcal{A}$ whose origin is at the center of mass. The vector $\tau$ is the torque applied the rigid body, considered as the input vector. The matrix $S(\Omega)$ is a skew-symmetric matrix such that $S(\Omega)V = \Omega \times V$ for any vector $V \in \mathbb{R}^3$, where $\times$ denotes the vector cross-product.

Our objective is to design a feedback controller, without velocity measurement, for the stabilization of the equilibrium point ($\tilde{R} := RR_d^T = I, \Omega - \Omega_d = 0$), where $R_d(t)$ is the desired orientation and $\Omega_d(t)$ is the desired angular velocity.

## III. Unit-Quaternion

The orientation of a rigid body with respect to the inertial frame can be described by a four-parameters representation, namely unit-quaternion [11]. A quaternion $Q = (q_0, q)$ is composed of a scalar component $q_0 \in \mathbb{R}$ and a vector $q \in \mathbb{R}^3$. The set of quaternion $\mathbb{Q}$ is a four-dimensional vector space over the reals, which forms a group with the quaternion multiplication denoted by "$\star$". The quaternion multiplication is distributive and associative but not commutative [11]. The multiplication of two quaternion $Q = (q_0, q)$ and $P = (p_0, p)$ is defined as [11], [13]

$$Q \star P = (q_0 p_0 - q^T p, q_0 p + p_0 q + q \times p) \tag{3}$$

and has the quaternion $(1, \mathbf{0})$ as the identity element. Note that, for a given quaternion $Q = (q_0, q)$, we have $Q \star Q^{-1} = Q^{-1} \star Q = (1, \mathbf{0})$, where $Q^{-1} = (q_0, -q)/\|Q\|^2$.

The set of unit-quaternion $\mathbb{Q}_u$ is a subset of $\mathbb{Q}$ such that

$$\mathbb{Q}_u = \left\{ Q = (q_0, q) \in \mathbb{R} \times \mathbb{R}^3 | q_0^2 + q^T q = 1 \right\}. \tag{4}$$

Note that in the case where $Q = (q_0, q) \in \mathbb{Q}_u$, the unit-quaternion inverse is given by $Q^{-1} = (q_0, -q)$.

A rotation matrix $R$ by an angle $\gamma$ about the axis described by the unit vector $\hat{k} \in \mathbb{R}^3$, can be described by a unit-quaternion $Q = (q_0, q) \in \mathbb{Q}_u$ such that

$$q = \hat{k} \sin\left(\frac{\gamma}{2}\right), \quad q_0 = \cos\left(\frac{\gamma}{2}\right) \tag{5}$$

The rotation matrix $R$ is related to the quaternion through the Rodriguez formula [7], [13]

$$\begin{aligned} R(Q) &= I + 2q_0 S(q) + 2S^2(q) \\ &= (q_0^2 - q^T q)I + 2qq^T + 2q_0 S(q). \end{aligned} \tag{6}$$

Algorithms allowing the extraction of $q$ and $q_0$ from a rotation matrix $R$, can be found in [13], [14].

In this note, instead of using the rotation matrix $R$ to describe the orientation of the rigid body, we will use the unit-quaternion. The dynamic equation (2) can be replaced by the following dynamic equation in terms of the unit-quaternion [7], [13]:

$$\dot{Q} = \frac{1}{2} Q \star Q_\Omega \tag{7}$$

where $Q = (q_0, q) \in \mathbb{Q}_u$ and $Q_\Omega = (0, \Omega) \in \mathbb{Q}$. In the sequel, we will use $Q_\star$ to denote the quaternion $(0, \star)$. We also define the unit-quaternion error $E = (e_0, e)$, which describes the discrepancy between two unit-quaternion $Q = (q_0, q)$ and $\bar{Q}(\bar{q}_0, \bar{q})$, as follows:

$$E = \bar{Q}^{-1} \star Q = (\bar{q}_0 q_0 + \bar{q}^T q,\ \bar{q}_0 q - q_0 \bar{q} - \bar{q} \times q). \tag{8}$$

Note that the unit-quaternion $Q$ and $\bar{Q}$ coincide if $E = (1, \mathbf{0})$.

It is also important to mention that the equilibrium point ($R = I, \Omega = 0$) for (1) and (2) is equivalent to the equilibrium point ($q = 0, q_0 = \pm 1, \Omega = 0$) for (1) and (7). Since $q_0 = 1$ corresponds to $\gamma = 0$ and $q_0 = -1$ corresponds to $\gamma = 2\pi$, it is clear that $q_0 = \pm 1$ correspond to the same physical point. Hence, the two equilibrium points ($q = 0, q_0 = \pm 1, \Omega = 0$) are in reality a unique physical equilibrium point corresponding to ($R = I, \Omega = 0$).

## IV. Main Results

Assume that the desired orientation to be tracked is given by

$$\dot{Q}^d = \frac{1}{2} Q^d \star Q_{\Omega_d} \tag{9}$$

where $\Omega_d$ is the desired angular velocity, which is assumed to be bounded as well as its first and second time-derivatives.

Let us define the unit-quaternion tracking error $Q^e$, which describes the discrepancy between the actual unit-quaternion $Q$ and the desired unit-quaternion $Q^d$, as follows: $Q^e = (Q^d)^{-1} \star Q := (q_0^e, q^e)$. Therefore, we have

$$Q^d \star Q^e = Q.$$

Differentiating both sides of the above equation with respect to time, we have

$$\dot{Q}^d \star Q^e + Q^d \star \dot{Q}^e = \dot{Q}.$$

Hence

$$\dot{Q}^e = (Q^d)^{-1} \star (\dot{Q} - \dot{Q}^d \star Q^e).$$

Using (7) and (9), the error quaternion dynamics is given by

$$\begin{aligned} \dot{Q}^e &= -\frac{1}{2} Q_{\Omega_d} \star Q^e + \frac{1}{2} Q^e \star Q_\Omega \\ &= -\frac{1}{2} Q^e \star (Q^e)^{-1} \star Q_{\Omega_d} \star Q^e + \frac{1}{2} Q^e \star Q_\Omega. \end{aligned} \tag{10}$$

Using the fact that $(Q^e)^{-1} \star Q_{\Omega_d} \star Q^e = Q_{\bar{\Omega}_d}$, with $\bar{\Omega}_d = R^T(Q^e)\Omega_d$, where $R^T(Q^e)$ is obtained from (6) by substituting $Q$ by $Q^e$, we have

$$\begin{aligned} \dot{Q}^e &= \frac{1}{2} Q^e \star Q_{\tilde{\Omega}} \\ &= \left( -\frac{1}{2} (q^e)^T \tilde{\Omega}, \frac{1}{2} (q_0^e I + S(q^e)) \tilde{\Omega} \right) \\ &:= (\dot{q}_0^e, \dot{q}^e) \end{aligned} \tag{11}$$

where $\tilde{\Omega} = \Omega - \bar{\Omega}_d$.

Let us introduce the following auxiliary system:

$$\dot{\bar{Q}} = \frac{1}{2} \bar{Q} \star Q_\beta \tag{12}$$

with $\bar{Q}(0) = (\bar{q}_0(0), \bar{q}(0)) \in \mathbb{Q}_u$, $Q_\beta = (0, \beta) \in \mathbb{Q}$, where the input $\beta$ of (12) will be designed later. We define the unit-quaternion $\tilde{Q} = \bar{Q}^{-1} \star Q^e = (\tilde{q}_0, \tilde{q}) \in \mathbb{Q}_u$ describing the discrepancy between the unit-quaternion tracking error $Q^e$ and the auxiliary unit-quaternion signal $\bar{Q}$.

Now, we can state the following theorem.

*Theorem 1:* Consider system (1) under the following control law

$$\tau = -\alpha_1 q^e - \alpha_2 \tilde{q} + I_f R^T(Q^e) \dot{\Omega}_d + S(\bar{\Omega}_d) I_f \bar{\Omega}_d \tag{13}$$

with $\alpha_1 > 0, \alpha_2 > 0$, and let the input of the auxiliary system (12) be

$$\beta = \Gamma_1 \tilde{q} \tag{14}$$

with $\Gamma_1 = \Gamma_1^T > 0$.

The vectors $q^e$ and $\tilde{q}$ are the vector parts of the unit-quaternion $Q^e$ and $\tilde{Q}$, respectively. Then, $Q^e$, $\tilde{Q}$ and $\Omega$ are globally bounded[2], and $\lim_{t\to\infty} q^e(t) = \lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} \tilde{\Omega}^\star(t) = 0$, $\lim_{t\to\infty} q_0^e(t) = \pm 1$ and $\lim_{t\to\infty} \tilde{q}_0(t) = \pm 1$, where $\tilde{\Omega}^\star(t) := \Omega(t) - \Omega_d(t)$.

*Proof:* The dynamical equation for the angular velocity tracking error is given by

$$I_f \dot{\tilde{\Omega}} = -(\tilde{\Omega} + \bar{\Omega}_d) \times I_f(\tilde{\Omega} + \bar{\Omega}_d) + I_f(\tilde{\Omega} \times \bar{\Omega}_d - R^T(Q^e)\dot{\Omega}_d) + \tau. \tag{15}$$

After some algebraic manipulations, one can show that

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\tilde{\Omega}^T I_f \tilde{\Omega}\right) = &-\tilde{\Omega}^T S(\bar{\Omega}_d) I_f \bar{\Omega}_d \\ &- \tilde{\Omega}^T (S(\bar{\Omega}_d) I_f + I_f S(\bar{\Omega}_d))\tilde{\Omega} \\ &+ \tilde{\Omega}^T(\tau - I_f R^T(Q^e)\dot{\Omega}_d).\end{aligned} \tag{16}$$

Since $I_f = I_f^T > 0$, it is clear that $(S(\bar{\Omega}_d)I_f + I_f S(\bar{\Omega}_d))$ is a skew symmetric matrix and hence $\tilde{\Omega}^T(S(\bar{\Omega}_d)I_f + I_f S(\bar{\Omega}_d))\tilde{\Omega} = 0$. Therefore

$$\frac{d}{dt}\left(\frac{1}{2}\tilde{\Omega}^T I_f \tilde{\Omega}\right) = \tilde{\Omega}^T(\tau - I_f R^T(Q^e)\dot{\Omega}_d - S(\bar{\Omega}_d)I_f\bar{\Omega}_d) \tag{17}$$

Using (11) and (12), one can show that

$$\begin{aligned}\dot{\tilde{Q}} &= \frac{d}{dt}(\bar{Q}^{-1} \star Q^e) \\ &= -\frac{1}{2}Q_\beta \star \tilde{Q} + \frac{1}{2}\tilde{Q} \star Q_{\tilde{\Omega}} \\ &= \left(\frac{1}{2}\tilde{q}^T(\beta - \tilde{\Omega}), \frac{1}{2}\tilde{q}_0(\tilde{\Omega} - \beta) + \frac{1}{2}\tilde{q} \times (\tilde{\Omega} + \beta)\right) \\ &:= (\dot{\tilde{q}}_0, \dot{\tilde{q}}).\end{aligned} \tag{18}$$

Consider the following Lyapunov function candidate:

$$\begin{aligned}V &= \alpha_2\left(\tilde{q}^T\tilde{q} + (\tilde{q}_0 - 1)^2\right) + \alpha_1\left((q^e)^T q^e + (q_0^e - 1)^2\right) \\ &\quad + \frac{1}{2}\tilde{\Omega}^T I_f \tilde{\Omega} \\ &= 2\alpha_2(1 - \tilde{q}_0) + 2\alpha_1(1 - q_0^e) + \frac{1}{2}\tilde{\Omega}^T I_f \tilde{\Omega}\end{aligned} \tag{19}$$

whose time-derivative, in view of (11), (17) and (18) is given by

$$\begin{aligned}\dot{V} &= -2\alpha_2\dot{\tilde{q}}_0 - 2\alpha_1\dot{q}_0^e + \frac{d}{dt}\left(\frac{1}{2}\tilde{\Omega}^T I_f \tilde{\Omega}\right) \\ &= -\alpha_2\tilde{q}^T(\beta - \tilde{\Omega}) + \alpha_1\tilde{\Omega}^T q^e \\ &\quad + \tilde{\Omega}^T(\tau - I_f R^T(Q^e)\dot{\Omega}_d - S(\bar{\Omega}_d)I_f\bar{\Omega}_d))\end{aligned} \tag{20}$$

which in view of (13) and (14), leads to

$$\dot{V} = -\alpha_2 \tilde{q}^T \Gamma_1 \tilde{q}. \tag{21}$$

Therefore, one can conclude that $\tilde{Q}$, $Q^e$ and $\tilde{\Omega}$ are globally bounded. Therefore, it is clear that $\ddot{V}$ is bounded. Hence, invoking Barbalat Lemma, one can conclude that $\lim_{t\to\infty} \tilde{q}(t) = 0$, which implies that $\lim_{t\to\infty} \tilde{q}_0(t) = \pm 1$. Consequently, one can show that $\ddot{\tilde{Q}}$ is bounded since $\dot{\Omega}_d$ is bounded, and hence $\lim_{t\to\infty} \dot{\tilde{Q}}(t) = 0$, which in turns, from (18), implies that $\lim_{t\to\infty}(\tilde{\Omega}(t) - \beta(t)) = 0$. Since $\lim_{t\to\infty} \tilde{q}(t) = 0$, it is clear, from (14), that $\lim_{t\to\infty} \beta(t) = 0$. Consequently, one can conclude that $\lim_{t\to\infty} \tilde{\Omega}(t) = 0$. Using the fact that $\ddot{\Omega}_d$ is bounded and the previous boundedness results, one can show that $\ddot{\tilde{\Omega}}$ is bounded, and hence, one can conclude that $\lim_{t\to\infty} \dot{\tilde{\Omega}}(t) = 0$. As $t$ goes to infinity, from (15), we have $0 = -I_f R^T(Q^e)\dot{\Omega}_d - S(\bar{\Omega}_d)I_f\bar{\Omega}_d + \tau$. Therefore, from (13), it is clear that $\lim_{t\to\infty}(\alpha_1 q^e(t) + \alpha_2\tilde{q}(t)) = 0$, which implies that $\lim_{t\to\infty} q^e(t) = 0$ since $\lim_{t\to\infty} \tilde{q}(t) = 0$. Finally, $\lim_{t\to\infty} q_0^e(t) = \pm 1$. Since $Q^e$ tends to $(\pm 1, 0)$, when t goes to infinity, it is clear $R(Q^e)$ goes to $I$ and hence, $\bar{\Omega}_d$ tends to $\Omega_d$. Consequently, $\lim_{t\to\infty}(\Omega(t) - \Omega_d(t)) = 0$. □

It is clear that our control scheme includes the attitude regulation problem as a particular case, i.e., $\Omega_d = 0$. The velocity-free attitude regulation scheme is given in the following Corollary.

*Corollary 1:* Consider system (1) under the following control law:

$$\tau = -\alpha_1 q^e - \alpha_2 \tilde{q} \tag{22}$$

$$\dot{\bar{Q}} = \frac{1}{2}\bar{Q} \star Q_\beta \tag{23}$$

$$\beta = \Gamma_1 \tilde{q} \tag{24}$$

where $\tilde{q}$ is the vector part of $\tilde{Q} = \bar{Q}^{-1} \star Q^e$, $\bar{Q}(0) \in \mathbb{Q}_u$, $Q_\beta = (0, \beta) \in \mathbb{Q}$, $\Gamma_1 = \Gamma_1^T > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$. Then, $\tilde{Q}$, $Q$ and $\Omega$ are globally bounded[3] and $\lim_{t\to\infty} q^e(t) = \lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} \Omega(t) = 0$, $\lim_{t\to\infty} \tilde{q}_0(t) = \pm 1$ and $\lim_{t\to\infty} q_0^e(t) = \pm 1$.

*Remark 1:* From the proof of Theorem 1, it is clear that for the closed loop system, $\dot{V} = 0$ at the following four equilibrium points $(\tilde{q}_0 = \pm 1, q_0^e = \pm 1, \tilde{\Omega}^\star = 0)$, and $\dot{V} < 0$ away from these equilibrium points. Note that these four equilibria represent the same physical equilibrium for the rigid body $(\tilde{R} := RR_d^T = I, \tilde{\Omega}^\star = 0)$. If initially, the closed-loop system is at one of these four equilibria, it will remain there for all subsequent time. In the case where the closed-loop system is not at one of the four equilibria, it will converge to the attractive equilibrium point $(\tilde{q}_0 = 1, q_0^e = 1, \tilde{\Omega}^\star = 0)$ for which $V = 0$ and $\dot{V} = 0$. The three isolated equilibrium points $(\tilde{q}_0 = 1, q_0^e = -1, \tilde{\Omega}^\star = 0)$, $(\tilde{q}_0 = -1, q_0^e = 1, \tilde{\Omega}^\star = 0)$ and $(\tilde{q}_0 = -1, q_0^e = -1, \tilde{\Omega}^\star = 0)$ are not attractors, but repeller equilibria [8].

*Remark 2:* The introduction of the auxiliary system (12) allows to generate a passive map $-\beta \mapsto \tilde{q}$[5]. In fact, this can be easily seen by substituting (13) in (20) to get

$$\int_0^T -\alpha_2 \tilde{q}^T \beta dt \geq V(X(T)) - V(X(0)) \tag{25}$$

with $X^T(t) = (\tilde{q}(t), \tilde{q}_0(t), q^e(t), q_0^e(t), \tilde{\Omega}(t))$. Therefore, the auxiliary system input $\beta$ can be designed in a straightforward manner as in (14). The resulting closed-loop system is a feedback interconnection of a passive system and a constant gain. This, guarantees global boundedness of $X(t)$ and the convergence of $\tilde{q}$ to zero. Finally, thanks to the fact that the largest positively invariant set $\{X | \dot{V} = 0\}$ is simply the set $\{X | \tilde{q} = 0, q^e = 0, \tilde{\Omega}^\star = 0\}$.

*Remark 3:* It is worth noting that the main purposes of the auxiliary dynamical system (12) are 1) to generate a passive mapping between $(-\beta)$ and the vector part of the unit quaternion error $\tilde{q}$; 2) to guarantee that the equilibrium of (18), in view of the fact that $\tilde{q}$ tends to zero, is characterized by $\tilde{\Omega} = \beta$. In fact, under the control law (13) and forcing the input $\beta$ of the auxiliary system (12) to be proportional to $\tilde{q}$, we ensure asymptotic convergence of $\tilde{q}$ to zero. The convergence of $\tilde{q}$ to zero will guarantee the convergence of $\tilde{\Omega}^\star$ to zero (as shown in the

[2]The global boundedness here indicates that the states are bounded for any $(Q^e(0), \tilde{Q}(0), \Omega(0)) \in \mathbb{Q}_u \times \mathbb{Q}_u \times \mathbb{R}^3$. Note that the unit-quaternion $\tilde{Q}$ and $Q^e$ are bounded by definition.

[3]The global boundedness here, means for any $(Q(0), \tilde{Q}(0), \Omega(0)) \in \mathbb{Q}_u \times \mathbb{Q}_u \times \mathbb{R}^3$.

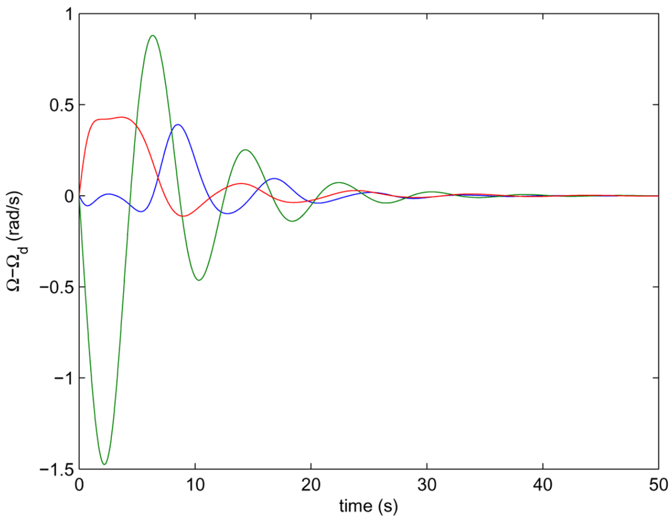


Fig. 1. The three components of the angular velocity error $\Omega - \Omega_d$ versus time.

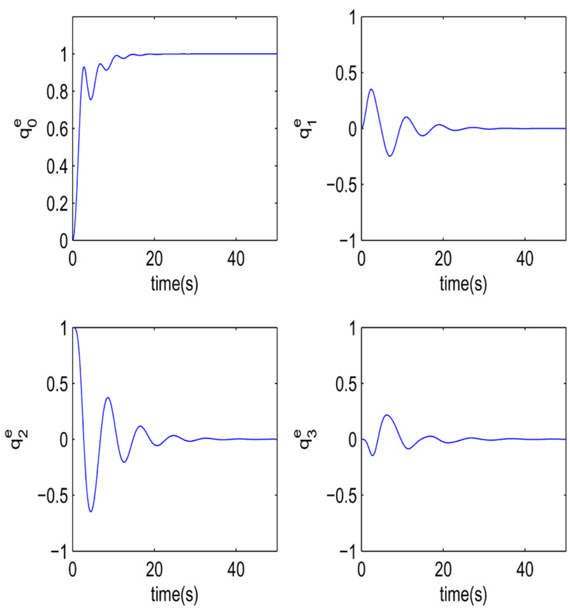


Fig. 2. Unit quaternion tracking error $Q^e = (q_0^e, q_1^e, q_2^e, q_3^e)$ versus time.

proof of Theorem 1) since the equilibrium point of (18), in view of the fact that $\tilde{q}$ tends to zero, is characterized by $\tilde{\Omega} = \beta$. Once $\tilde{q}$ and $\tilde{\Omega}^\star$ converge to zero, the convergence of $q^e$ to zero is guaranteed in view of the system dynamics (1) and the structure of the control law (13).

*Remark 4:* In the regulation case, our control law (22) is a pure quaternion feedback (since $q^e$ and $\tilde{q}$ are the vector parts of unit quaternion). Since $\|q^e\| \leq 1$ and $\|\tilde{q}\| \leq 1$, it is clear that the control effort is bounded as $\|\tau\| \leq \alpha_1 + \alpha_2$ and hence a natural saturation, interms of the control gains, is achieved and the designer can set the limits of the control effort through the control gains $\alpha_1$ and $\alpha_2$. This conclusion cannot be achieved with the regulation controller of [10] since the term substituting the angular velocity is not a unit-quaternion and is frequency dependant.

## V. Simulation Results

In this section, we present some simulation results showing the effectiveness of the proposed controller. The inertia matrix has been taken as

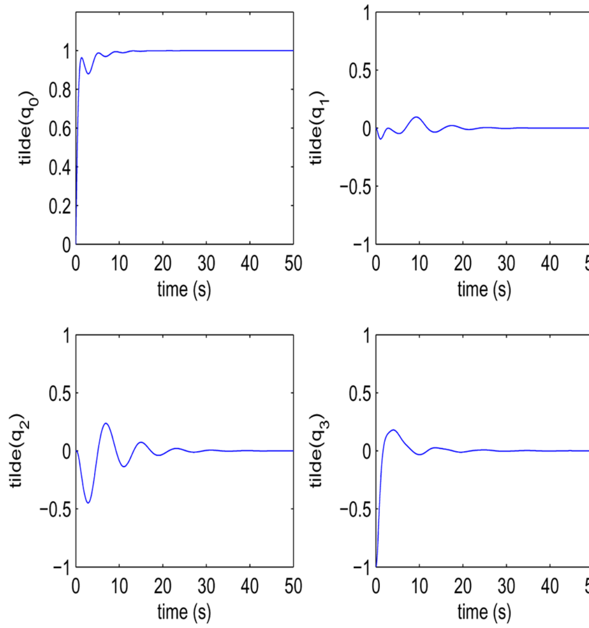


Fig. 3. $\tilde{Q} = (\tilde{q}_0, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ versus time.

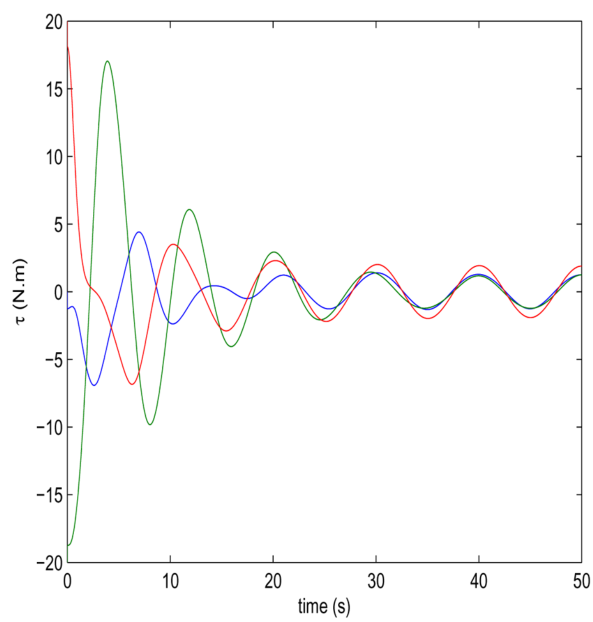


Fig. 4. Control input $\tau$ versus time.

$I_f = \mathrm{diag}(20, 20, 30)$. We applied the control law of Theorem 1, with $\alpha_1 = \alpha_2 = 20$ and $\Gamma_1 = \mathrm{diag}(3, 3, 3)$. The initial conditions have been taken as follows: $Q(0) = (0, 0, 1, 0)$ and $\bar{Q}(0) = (0, 1, 0, 0)$. The reference trajectory is given by (9) with $Q^d(0) = (1, 0, 0, 0)$ and $\Omega_d = 0.1\sin(0.2\pi t)[1, 1, 1]^T$. The simulation was performed with Simulink for a time span of 50 s.

Fig. 1 shows the evolution of the three components of the angular velocity tracking error $\Omega - \Omega_d$ with respect to time. Fig. 2, shows the evolution of the unit-quaternion tracking error $Q^e$, describing the deviation between the orientation of the body and the desired orientation, with respect to time. Fig. 3, shows the time evolution of the unit-quaternion error $\tilde{Q}$, describing the deviation between $Q^e$ and $\bar{Q}$. Fig. 4, shows the control input versus time.

## VI. Conclusion

A new quaternion-based solution to the attitude tracking problem, without velocity measurement, has been proposed. Our approach is based on the use of a unit-quaternion auxiliary system whose input is related to the vector part of the unit quaternion error $\tilde{q}$ via a passive map, under an appropriate unit quaternion-based feedback. The proposed control scheme includes the attitude regulation problem as a particular case, and guarantees *almost* global asymptotic stability of the equilibrium point ($\tilde{R} := RR_d^T = I, \tilde{\Omega}^\star = 0$). In the regulation case, our control scheme is a pure quaternion feedback, and consequently, the designer can set, in a straightforward manner, the upper bound for the control effort in terms of the control gains.

## References


[1] M. R. Akella, J. T. Halbert, and G. R. Kotamraju, "Rigid body attitude control with inclinometer and low-cost gyro measurements," *Syst. Contr. Lett.*, vol. 49, pp. 151–159, 2003.

[2] S. P. Bhat and D. S. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Syst. Contr. Lett.*, vol. 39, pp. 63–70, 2000.

[3] F. Caccavale and L. Villani, "Output feedback control for attitude tracking," *Syst. Contr. Lett.*, vol. 38, no. 2, pp. 91–98, 1999.

[4] B. T. Costic, D. M. Dawson, M. S. d. Queiroz, and V. Kapila, "A quaternion-based adaptive attitude tracking controller without velocity measurements," *AIAA J. Guidance, Contr. Dynam.*, vol. 24, no. 6, 2001.

[5] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, ser. Electrical Science . New York: Academic, 1975.

[6] O. Egeland and J. M. Godhavn, "Passivity-based adaptive attitude control of a rigid spacecraft," *IEEE Trans. Autom. Control*, vol. 39, pp. 842–846, 1994.

[7] P. C. Hughes, *Spacecraft Attitude Dynamics*. New York: Wiley, 1986.

[8] S. M. Joshi, A. G. Kelkar, and J. T.-Y. Wen, "Robust attitude stabilization of spacecraft using nonlinear quaternion feedback," *IEEE Trans. Autom. Control*, vol. 40, pp. 1800–1803, 1995.

[9] D. E. Koditschek, "Application of a new Lyapunov function to global adaptive attitude tracking," in *Proc. 27th IEEE CDC*, Austin, TX, 1988, pp. 63–68.

[10] F. Lizarralde and J. T. Wen, "Attitude control without angular velocity measurement: A passivity approach," *IEEE Trans. Autom. Control*, vol. 41, pp. 468–472, 1996.

[11] R. M. Murray, Z. Li, and S. Satry, *A Mathematical Introduction to Robotic Manipulation*. Baton Raton, FL: CRC, 1994.

[12] S. Salcudean, "A globally convergent angular velocity observer for rigid body motion," *IEEE Trans. Autom. Control*, vol. 36, pp. 1493–1497, 1991.

[13] M. D. Shuster, "A survey of attitude representations," *J. Astronaut. Sci.*, vol. 41, no. 4, pp. 439–517, 1993.

[14] W. Stanley, "Quaternion from rotation matrix," *J. Guidance Control*, vol. 1, no. 3, pp. 223–224, 1978.

[15] A. Tayebi and S. McGilvray, "Attitude stabilization of a quadrotor aircraft," *IEEE Trans. Control Syst. Technol.*, vol. 14, pp. 562–571, 2006.

[16] A. Tayebi, "Unit quaternion observer based attitude stabilization of a rigid spacecraft without velocity measurement," in *In proc. of the 45th IEEE Conference on Decision and Control*, San Diego, CA, 2006, pp. 1557–1561.

[17] P. Tsiotras, "Further results on the attitude control problem," *IEEE Trans. Autom. Control*, vol. 43, pp. 1597–1600, 1998.

[18] J. T.-Y. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Trans. Autom. Control*, vol. 36, pp. 1148–1162, 1991.

[19] B. Wie, H. Weiss, and A. Arapostathis, "Quaternion feedback regulator for spacecraft eigenaxis rotations," *AIAA J. Guidance Control*, vol. 12, no. 3, pp. 375–380, 1989.


# Robust Control of Nonlinear Jump Parameter Systems Governed by Uncertain Chains

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***Abstract*—We consider an infinite-horizon minimax optimal control problem for stochastic uncertain systems governed by a discrete-state uncertain continuous-time chain. Using existing risk-sensitive control results, a robust suboptimal absolutely stabilizing guaranteed cost controller is constructed. Conditions are presented under which this suboptimal controller is minimax optimal. We then present a numeric algorithm for calculating a robust (sub)optimal controller using a Markov chain approximation technique.**

***Index Terms*—Markov chain approximations, Markov jump parameter systems, robust control, stochastic control.**

## I. Introduction

Minimax robust control of uncertain stochastic systems, in which perturbations are restricted to satisfy a constraint on probability laws associated with disturbances, has been actively developed in the past decade [1]–[3]. This theory covers problems of robust LQG control and filtering, and also nonlinear control systems [1], [3], controllability, observability, and performance aspects of robust controllers and filters [4], [5]. The theory is however limited in that it only applies to systems subject to Gaussian disturbances. In this paper, we expand the boundaries of this theory to include nonlinear hybrid stochastic systems governed by a discrete-state uncertain mode process. In addition, dynamics of each mode of the system are subject to disturbances.

The problem in the focus of this paper is that of nonlinear robust switching control design via optimization of the worst-case performance of an uncertain stochastic system driven by an uncertain noise and subject to abrupt changes of system parameters. We wish to find a state-feedback switching control solution $u^*$ to the worst-case performance optimization problem

$$\inf_u \sup_{Q\in\Xi_d} J(u,Q) \leq \sup_{Q\in\Xi_d} J(u^*,Q),$$
$$J(u,Q) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{E}^{Q^T} c(x(t),u(t),\mathbf{r}(t))dt. \quad (1)$$

Here, $x(t)$ is the state process and $\mathbf{r}$ describes a discrete-event random mechanism of mode changes. Both processes evolve under an uncertain probability measure $Q$, and have uncertain probability distributions subject to the constraint $Q \in \Xi_d$; $\Xi_d$ is a given set. We refer to Section II for rigorous definitions. A controller sought is allowed to access both $x$ and $\mathbf{r}$.

The major novelty of this paper is the 'hybrid' uncertainty model which combines the uncertainties in the discrete-event and continuous-state components of the system. Indeed, in a hybrid system, plant modeling errors may depend on the state of the mode process. Also, probabilities of switching from one operation mode to another mode may de-

Manuscript received June 28, 2005; revised June 5, 2007. Current version published August 29, 2008. Recommended by Associate Editor H. Hjalmarsson. This work was supported by the Australian Research Council. Part of this work was done during the second author's visit to the Australian National University.

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Digital Object Identifier 10.1109/TAC.2008.928911